Practice

§7.5: 1,4,7,13,17,25
§7.6: 3,5,7,9,25
§7.8: 3,5,9
§7.9: 1,3,5

Hand In

(You may use MAPLE (or similar) to compute eigenvalues and eigenvectors. However you may not use the `MatrixExponential` function (or similar) to compute $\exp(At)$ without any further work.

54) Solve the initial value problem

$$\underline{\mathbf{x}}'(t) = \begin{pmatrix} 10 & -53 \\ 10 & 12 \end{pmatrix} \cdot \underline{\mathbf{x}}(t), \quad \underline{\mathbf{x}}(0) = \begin{pmatrix} 23 \\ 23 \end{pmatrix}.$$

55) Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

a) Show that 3 is the only eigenvalue for A and that it has deficiency 2
b) Calculate $\exp(A \cdot t)$.

56*) Determine a real-valued fundamental set of solutions for the differential equation

$$\underline{\mathbf{x}}' = \begin{pmatrix} -9 & 108 & -92 \\ 10 & -90 & 79 \\ 12 & -124 & 107 \end{pmatrix} \cdot \underline{\mathbf{x}}$$

57) Determine a general solution to the following system of differential equations:

$$\underline{\mathbf{x}}' = \begin{pmatrix} -723 & 280 & 2744 \\ -91 & 40 & 343 \\ -182 & 70 & 691 \end{pmatrix} \cdot \underline{\mathbf{x}} + \begin{pmatrix} e^{5t} \\ 3t+7 \\ 20 \end{pmatrix}$$

58*) Determine a solution to the initial value problem $\underline{\mathbf{x}}' = A \cdot \underline{\mathbf{x}}$ with

$$A := \begin{pmatrix} 1 & 3 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 18 & -5 & 0 & 6 & -25 & 0 & -2 \\ 0 & 64 & -18 & 0 & 24 & -100 & 0 & -8 \\ -2 & 6 & 0 & 2 & 0 & -1 & -2 & 0 \\ 0 & 116 & -34 & 0 & 43 & -170 & 0 & -14 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 45 & -13 & 0 & 15 & -65 & 1 & -5 \\ 0 & 322 & -94 & 0 & 114 & -470 & 0 & -37 \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Problems marked with a *are bonus problems for extra credit.

Computing the Jordan Canonical Form of a matrix

Definition If M is a matrix, then $\mathcal{N}(M)$ is the set of vectors $\underline{\mathbf{x}}$ such that $M\underline{\mathbf{x}} = \underline{\mathbf{0}}$. A basis of $\mathcal{N}(M)$ (a linearly independent set that can generate all elements) is obtained by triangulizing (RREF) M and selecting all choices of one freely choosable variable to 1 and all other choosable variables to 0. The dimension $\dim(\mathcal{N}(M))$ is the number of vectors obtained this way.
If $\{\underline{\mathbf{x_1}}, \ldots, \underline{\mathbf{x_n}}\}$ is a set of vectors, we write

$$\left\langle \underline{\mathbf{x_1}}, \ldots, \underline{\mathbf{x_n}} \right\rangle = \left\{ \sum_{i=1}^{n} c_i \underline{\mathbf{x_i}} \mid c_i \in \mathbb{C} \right\}$$

for the set of all linear combinations of these vectors. This set is called the span of these vectors.
The following algorithm shows how one can compute the Jordan canonical form of a matrix. We assume that the characteristic polynomial splits into linear factors (for example, if we work over the complex numbers). To see that the algorithm works as desired, consider what happens to a jordan block A for eigenvalue λ if we look at the nullspaces of $(A - \lambda \cdot I)^n$.

(1) Determine the Eigenvalues λ_i of the matrix A (for example as roots of the characteristic polynomial). For each eigenvalue λ perform the following calculation (which gives a basis for the generalized eigenspace of λ, the whole basis will be obtained by concatenating the bases obtained for the different λ_i):

(2) Calculate $n_i = \dim \mathcal{N}((A - \lambda)^i)$ until the sequence becomes stationary. (The largest n_i is the dimension of the generalized eigenspace, i.e. the number of basis vectors corresponding to λ.)

(3) Let $d_i = n_i - n_{i-1}$. (One can show that $d_i - d_{i+1}$ is the number of $i \times i$ Jordan blocks.)

4) We now build a basis in sequence of descending i. Let $B = [\,]$ and $i = \max\{i \mid d_i > 0\}$. ($B$ will become a list of lists, each list corresponding to one Jordan block.)

(5) For each list $[s1, \ldots, s_m]$ in B, append the image $A \cdot s_m$ of its last element to this list.

(6) If $d_i - d_{i+1} = m > 0$, determine m linearly independent vectors $b_1, \ldots b_m$ in $\mathcal{N}((A-\lambda)^i$ which are not already in $\mathcal{N}((A-\lambda)^{i+1})$, nor generated by the union (over all lists) of the s_i.

The probability is very high that any (random) m linear independent basis vectors of $\mathcal{N}((A-\lambda)^i)$ fulfill this property. Pick m such vectors and verify that they are not in the subspace generated by all the s_j.

(The generic method would be to extend the basis of the subspaces to a basis of $\mathcal{N}((A-\lambda)^i)$ and take the vectors by which one extended the basis.)

(7) For each such vector b_j obtained this way, add a list $[b_j]$ to B. (i.e. we start a new Jordan block of size i.)

(8) If the number of vectors in the lists in B is smaller than the maximal n_i, then decrement i and go to step (5).

(9) Concatenate the reverses of the lists in B. This is the part of the basis corresponding to eigenvalue λ.

For example, let

$$A := \begin{pmatrix} 59 & -224 & 511 & -214 & 4 \\ 16 & -61 & 139 & -58 & 1 \\ 6 & -24 & 51 & -20 & 0 \\ 13 & -52 & 110 & -43 & 0 \\ -4 & 12 & -38 & 20 & -1 \end{pmatrix}.$$

Its characteristic prolynomial is $(x-1)^4$. We get the following nullspace dimensions and their differences:

i	0	1	2	3	4
$e_i = \dim(\mathcal{N}((A-1)^i))$	0	2	4	5	5
$f_i = e_{i+1} - e_i$	–	2	2	1	0

At this point we know already the shape of the Jordan Canonical form of A: There is one 3×3 block and one 2×2 block.
Let us compute the explicit basis:
We start at $i = 3$ and set $B = [\,]$. We have that $\mathcal{N}((A-1)^3) = \mathbb{R}^5$ and

$$\mathcal{N}((A-1)^2) = \left\langle (4,1,0,0,0)^T, (-15/2,0,1,0,0)^T, (3,0,0,1,0)^T, (0,0,0,0,1)^T \right\rangle,$$

We thus pick $b_1 := (1,0,0,0,0)^T$ as first basis vector (an almost random choice, we only have to make sure it is not contained in $\mathcal{N}((A-1)^2)$, which is easy to verify) and add the list $[b_1]$ to B.

In step $i = 2$ we first compute the image $b_2 := (A-1)b_1 = (58, 16, 6, 13, -4)^T$ and add it to the list.
Furthermore, as $f_2 > f3$, we have to get another basis vector in $\mathcal{N}((A-1)^2)$, but not in $\mathcal{N}((A-1)^3)$, nor in the span of b_2. We pick $b_3 = (4, 1, 0, 0, 0)^T$ from the spanning set of $\mathcal{N}((A-1)^2)$, and verify that it indeed fulfills the conditions. We thus have $B = [[b_1, b_2], [b_3]]$.
In step $i = 1$ we now compute images again $b_4 := (A-1)b_2 = (48, 12, 4, 10, 0)^T$ and (from the second list) $b_5 := (8, 2, 0, 0, -4)^T$.
As $f_1 = f_2$ no new vectors are added.
As a result we get $B = [[b_1, b_2, b_4], [b_3, b_5]]$.
Finally we concatenate the reversed basis vector lists and get the new basis $(b_4, b_2, b_1, b_5, b_3)$.
We thus have the base change matrix

$$M := \begin{pmatrix} 48 & 58 & 1 & 8 & 4 \\ 12 & 16 & 0 & 2 & 1 \\ 4 & 6 & 0 & 0 & 0 \\ 10 & 13 & 0 & 0 & 0 \\ 0 & -4 & 0 & -4 & 0 \end{pmatrix}. \quad \text{and} \quad M^{-1}AM = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is in Jordan Canonical Form.

Summary

To solve a system $\underline{\mathbf{x}}' = A\underline{\mathbf{x}} + (\underline{g})$ of differential equations for constant A.

1. If $g = 0$, the solution has the form $\underline{\mathbf{x}}(t) = \exp(At)\underline{\mathbf{x_0}}$ and the columns of $\exp(At)$ form a fundamental set of solutions. To calculate $\exp(At)$
   (a) Calculate eigenvalues and eigenvectors of A.
   (b) If no eigenvalue is deficient, let M be the matrix whose columns are formed from bases of the eigenvectors.
   (c) Then $D = M^{-1}AM$ is diagonal with the eigenvalues on the diagonal.
   (d) Calculate $\exp(Dt)$ (which is easy, just a diagonal matrix with $e^{\lambda_i t}$ on the diagonal). Then $\exp(At) = M\exp(Dt)M^{-1}$.
   (e) If eigenvalues are deficient calculate the Jordan canonical form J of A and a transforming matrix M. Again $\exp(At) = M\exp(Jt)M^{-1}$
   (f) $\exp(Jt)$ can be calculated from the definition $\exp(J) = \sum_{i=0}^{\infty} \frac{1}{i!}J^i$: If there is an $l \times l$ Jordan block, $\exp(Jt)$ has diagonal entries $e^{\lambda t}$, off diagonal entries $te^{\lambda t}$, next off diagonal $\frac{1}{2}t^2e^{\lambda t}$ etc. up to the $l-1$ diagonal.
2. If $g \neq 0$ and A is diagonalizable with eigenvector matrix M (i.e. $D = M^{-1}AM$ is diagonal. Set $\underline{\mathbf{y}} = M^{-1}\underline{\mathbf{x}}$. Then the system is equivalent to $\underline{\mathbf{y}}' = D\underline{\mathbf{y}} + (M^{-1}\underline{\mathbf{g}})$, which has separated variables and initial value $\underline{\mathbf{y}}(t_0) = M^{-1}\underline{\mathbf{x}}(t_0)$. The solution is $\underline{\mathbf{x}}(t) = M\underline{\mathbf{y}}(t)$.