30] Suppose that $\Gamma$ is not connected. Then $\Gamma$ has at least two components. Let $v, w \in U$.
If $v, w$ are in <u>different</u> components of $\Gamma$, then they are not connected in $\Gamma$, and thus connected in $\overline{\Gamma}$.
If they are in the same component of $\Gamma$, let $x \in U$ be in a different component. Thus $\{v,x\}, \{w,x\} \notin E$ in $\Gamma$ and thus are connected in $\overline{\Gamma}$. Thus
$(v, \{v,x\}, x, \{w,x\}, w)$ is a walk in $\overline{\Gamma}$, so $\overline{\Gamma}$ is connected.

32) We have $(I+A)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A^k$ by the binomial theorem. As the entries in $A^k$ are nonnegative and as the coefficients $\binom{n-1}{k}$ are nonnegative, then $(I+A)^{n-1}$ has nonzero entries in a position $i,j$ iff for some $k$ $A^k$ has a nonzero entry in this position. But that is the case iff there is a walk from $i$ to $j$,
This holds for all $i,j$ iff $G$ is connected.

33) ~~Suppose all vertices have different degree. Then the number of edges is $\frac{1}{2}\sum_{v \in U} \deg(v)$~~

33] Assume all $n$ vertices have different degrees, they must be chosen from $\{\underbrace{0, \dots, n-1}_{n \text{ values}}\}$.
If $G$ is connected no vertex has degree zero, but $\{1, \dots, n-1\}$ has only $n-1$ distinct values.

If $G$ is not connected no vertex may have degree $n-1$ (as it otherwise would connect all others), but $\{0, \dots, n-2\}$ has only $n-1$ distinct values.

34) If $(x=v_0, e_1, v_1, e_2, \ldots\ v_{k-1}, e_k, v_k=y)$

is a walk from x to y, then (let $f(e)=\{f(z) / z\in e\}$)

$(f(x), f(e_1), f(v_1), \ldots\ , f(e_k), f(y))$

is a walk from $f(x)$ to $f(y)$. Thus

$d(f(x), f(y)) \leq d(x,y)$. Using the inverse of f (which is also an isomorphism) shows the reverse inequality

35) 1) d counts the length of a path and thus is nonnegative. If $d(x,y)=0$, the shortest path has length 0 and thus $x=y$.

2) Consider the reversed path.

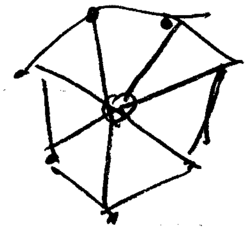
3) If $(x, e_1, v_1, \ldots\ , v_{k-1}, e_k, y)$ is a min length path from x to y and $(y, f_1, w_1, \ldots\ , w_{\ell-1}, f_\ell, z)$ a min length path from y to z then

$(x, e_1, v_1, \ldots, v_{k-1}, e_k, y, f_1, w_1, \ldots, f_\ell, z)$

is a walk ~~of~~ from x to z. Thus

$d(x,z) \leq k+\ell = d(x,y)+d(y,z)$.

36]

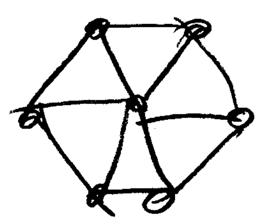


37] $(100, 0, 0, 0)$, as one vertex cannot have positive degree if no other has.

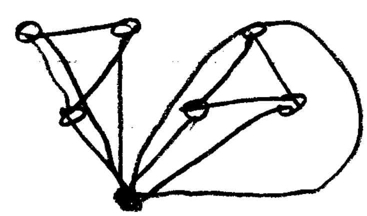
38] Using theorem 4.3.3:

$(4, 4, 4, 2, 2)$
$\to (3, 3, 1, 1)$
$\to (2, 0, 0)$ cannot be degree sequence.

39) The vertex of degree 6 must connect to all other vertices. These remaining vertices all have degree 2 and thus are the induced subgraph is a union of cycles. Thus



or



40) Test for any $1 \le r \le n-1$ that

$$\sum_{i=1}^{r} d_i \le r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i)$$

1:

| r | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| LHS | 4 | 7 | 10 | 13 | 15 | 17 | 19 |
| RHS | 7 | 13 | 16 | 19 | 25 | 33 | 43 |

this is degree sequence

2: For $r=3$ LHS is 21, RHS is 20

this not degree sequence

3: Yes

4: Yes.