16. Formula d) involves no division or subtraction and thus never needs to handle numbers larger than the result

17. a) If $(i,j) \in S$ then $i \neq j$, and thus $(j,i) \neq (i,j)$, ~~then~~ and ~~for~~ either $i<j$ or $j<i$. Thus $|S|$ is ~~the~~ ~~[illegible]~~ twice $|\{(i,j) \in S \mid i<j\}|$, and thus even.

b) Let x_i be the number person i shakes hands. Then $x_i = |\{(a,b) \in S \mid a = i\}|$. These sets partition S, thus $\sum_i x_i = |S|$

c) As $|S|$ is even, $\sum_i x_i$ is even. This can only happen if an even number of x_i's are odd

18| Let $\{a_1, \ldots, a_k\} \subseteq \{1 \ldots n\}$ such that

$a_1 < a_2 \cdots < a_k$ and not consecutive

$\Rightarrow a_{i+1} \geq a_i + 2$.

Claim 1: $a_i - i + 1 > a_{i-1} - (i-1) + 1 = a_{i-1} - i + 2$

because $a_i - i + 1 \geq a_{i-1} - i + 1 + 2 > a_{i-1} - i + 2$

Thus $\{a_1, a_2 - 1, \ldots, a_k - k + 1\}$ is a ~~strictly increasing~~ set of k numbers

with $a_1 < a_2 - 1 < \cdots < a_k - k + 1$.

Claim 2: Since $a_k \leq n$ we have that $a_k - k + 1 \leq n - k + 1$

Thus the map goes to subsets of $\{1, \ldots, n-k+1\}$

Claim 3: The map is invertible, as the inverse is

$\{b_1, \ldots, b_k\} \mapsto \{b_1, b_2 + 1, b_3 + 2, \ldots, b_k + k - 1\}$

and if $b_i \gneq b_{i-1}$ then $b_i + i - 1 \gneq b_{i-1} + i - 1 - 1$.

Thus bijection and the number is equal to that of subsets of $\{1, 2, \ldots, n-k+1\}$ which is

$$\binom{n-k+1}{k}$$

19| Done in class

20] Proof by Induction.

Base case: $n=0$: $(fg)^{(0)} = fg = \sum_{k=0}^{0} \binom{0}{0} f^{(0)} g^{(0)}$

Inductive step: Assume true for n. Then

$$(f g)^{(n+1)} = \frac{d}{dx}\left((f g)^{(n)}\right) \overset{\text{Induction}}{=} \frac{d}{dx}\left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx}\left(f^{(k)} g^{(n-k)}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}\right)$$

let $j = k+1$

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} \left(f^{(j)} g^{(n-j+1)}\right) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)}$$

allowing 0 does not change as $\binom{n}{-1} = 0$

allowing $n+1$ does not change as $\binom{n}{n+1} = 0$

$$= \sum_{k=0}^{n+1} \underbrace{\left(\binom{n}{k-1} + \binom{n}{k}\right)}_{=\binom{n+1}{k}} \left(f^{(k)} g^{(n-k+1)}\right) \text{ as desired.}$$