Mathematics 301 **Midterm 2 (100 points)** 11/10/17

Points (leave blank)

1	2	3	4	5	Σ

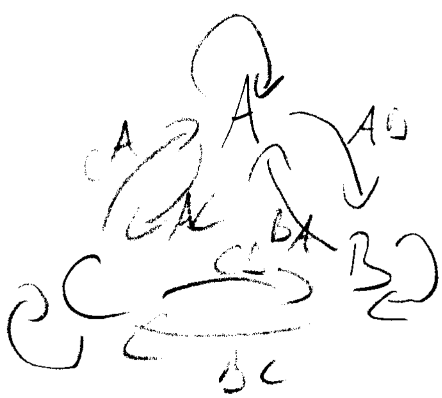
Name:
(clearly, please)

Honor pledge: I have not given, received, or used any unauthorized assistance.

Signature

You may use a pocket calculator that is incapable of transmitting data; it may not store any user-defined information. You also may bring a handwritten single page, letter size with notes. You can work on the problems in any order you like. Show your work! All problems carry the same weight. Calculation steps and explanation for statements made are a crucial part of a solution. Partial credit will be given sparingly – rather complete one problem than start two only partially.

1) Construct a cyclic sequence of length 9, such that any 2-element sequence of the letters A, B, C occurs as subsequence.



Eulerian PATH:

A BCAACCBB

2) Which of the following sequences can occur as degree sequence of a graph? Either give an example of such a graph, or prove that no such graph may exist.

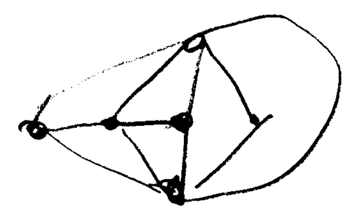
$$a)(6,5,4,3,3,2,1) \qquad b)(6,6,5,4,3,3,1)$$

a) $(6,5,4,3,3,2,1)$ Score theorem

$(4,3,2,2,1,0) \Rightarrow (4,3,2,2,1)$

$(2,1,0)$ exists

by the score theorem this $(6,5,4,3,3,2,1)$ is a degree sequence. Example:



b) $(6,6,5,4,3,3,1)$

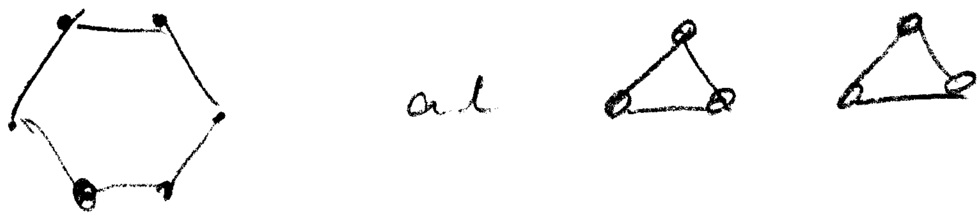
$(5,4,3,2,2,0) \rightarrow (5,4,3,2,2)$

cannot subtract 5 here as only 4 entries (respectively entry 0)

By the score theorem this is not a valid degree sequence

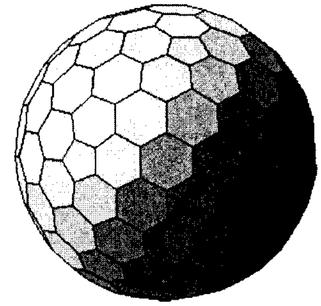
3) Does the degree sequence of a graph determine whether the graph is connected? If true give a proof, if false give a concrete counterexample (which will consist of two graphs, one connected, one disconnected, with the same degree sequences).

No: The graphs



both have degree sequence $(2,2,2,2,2,2)$, but one is connected, the other one not.

4) A *geode* is a sphere-like structure whose surface is composed from pentagons and hexagons (such as in the picture on the side). At every corner exactly 3 faces meet. Show that a geode must have exactly 12 pentagons. (**Hint:** Assume there are a pentagons and b hexagons, calculate the number of endges, vertices and faces and use Euler's polyhedron theorem.)



$F = a+b$

$E = \frac{5a+6b}{2}$ (five, respectively 6 edges per face, but every edge has 2 faces adjoint)

$V = \frac{5a+6b}{3}$ (ditto, every vertex is on 3 faces)

Euler:

$$2 = V - E + F = \frac{3a+6b}{3} - \frac{5a+6b}{3} + a + b$$

$$= \frac{a + 0b}{6}$$ Thus there are

$a = 12$ pentagons.

5) Let $G = (V, E)$, $H = (W, D)$ be graphs and $f: V \to W$ an isomorphism of the two graphs.
a) Show that if

$$(v_0, e_1 = \{v_0, v_1\}, v_1, e_2 = \{v_1, v_2\}, v_2, \ldots, v_{k-1}, e_k = \{v_{k-1}, v_k\}, v_k)$$

is a path in G that

$$(f(v_0), f(e_1) = \{f(v_0), f(v_1)\}, f(v_1), \ldots, f(v_{k-1}), f(e_k) = \{f(v_{k-1}), f(v_k)\}, f(v_k))$$

is a path in H.
b) Show that the diameter of G (the maximum length of a shortest path between two vertices) must be equal to the diameter of H.

a) In the given image sequence, we have a sequence of vertices and edges between vertices. ~~Assume this~~ Thus it is a walk.
Assume it is not a path, then $f(v_i) = f(v_j)$ for some $i \neq j$. but then (as f is one-to-one) we must have $v_i = v_j$, a contradiction to the first walk being a path.

b) From a it follows that if $d(v_0, v_k) = k$ there is a path in H from $f(v_0)$ to $f(v_k)$ of length k thus $d_G(v_0, v_k) \geq d_H(f(v_0), f(v_k))$
Using the inverse of f we get ditto:

$$d_H(f(v_0), f(v_k)) \geq d_G(\underbrace{f^{-1}(f(v_0))}_{= v_0}, \underbrace{f^{-1}(f(v_k))}_{= v_k})$$

and thus equality of distances.

The diameter is the maximum distance
and thus must be equal amongst the isomorphic
graphs aswell.