Mathematics 301							**Midterm 1 (100 points)**	9/29/17
Points (leave blank)								
1	2	3	4	5	Σ	Name: (clearly, please)	SAMPLE	

Honor pledge: I have not given, received, or used any unauthorized assistance.

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Signature

You may use a pocket calculator that is incapable of transmitting data; it may not store any user-defined information. You also may bring a handwritten single page, letter size with notes. You can work on the problems in any order you like. Show your work! All problems carry the same weight. Calculation steps and explanation for statements made are a crucial part of a solution. Partial credit will be given sparingly – rather complete one problem than start two only partially.

1) Consider an $m \times n$ chessboard. You place a figure on the bottom left corner and want to move it to the top right corner. Each move is either one step right or one step up. In how many different ways can you do this? Give a formula involving m and n.

There are $m-1$ moves up and $n-1$ moves right. We just need to indicate which of the $m-1+n-1=m+n-2$ moves are right: $\binom{m+n-2}{n-1}$ possibilities

2) a) Expand $(1+x)^7$. given by binomial coeffs / Pascal's triangle

$$1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

b) Let $a = e^{\frac{2\pi i}{35}}$ (as a complex number), that is $a^{35} = 1$ (You do not need to show this). Evaluate the sum $\sum_{i=0}^{34} a^i$, i.e. give the result as a complex number.

By geom. series: $$\sum_{i=0}^{34} a^i = \frac{a^{34+1}-1}{a-1} = \frac{a^{35}-1}{a-1}$$

$$= \frac{0}{a-1} = 0$$

as $a^{35} = 1$

3) How many different 4-letter sequences can you form from the letters of the word MOUNTAIN? Give the result as a number.

There are 7 letters (MOUNTAI) w/o duplicates

Thus, if no letter repeats, there are

$7 \cdot 6 \cdot 5 \cdot 4 = 840$ words.

If we have a duplicate letter, it is N.

There are $\binom{4}{2} = 6$ possibilities to place the

N's and then $6 \cdot 5$ possibilities for the remaining

2 letters. ~~total~~ Thus: $6 \cdot 6 \cdot 5 = 180$

In total: $840 + 180 = 1020$ words

4) How many numbers from 0 to 1000 are multiples of neither 5, 11, or 13?

Let A_1 = multiples of 5

A_2 = multiples of 11

A_3 = multiples of 13

Then $|A_1| = \left\lceil \frac{1001}{5} \right\rceil = 201$ $\quad |A_3| = 77$

$|A_2| = \left\lceil \frac{1001}{11} \right\rceil = 91$ $\quad |A_1 \cap A_2| = \left\lceil \frac{1001}{55} \right\rceil = 19$

$|A_1 \cap A_3| = 16$, $|A_2 \cap A_3| = 7$, $|A_1 \cap A_2 \cap A_3| = 2$

Thus
$$\left|\bigcup A_i\right| = \underset{201}{|A_1|} + \underset{91}{|A_2|} + \underset{77}{|A_3|} - \underset{19}{|A_1 \cap A_2|}$$
$$- \underset{16}{|A_1 \cap A_3|} - \underset{7}{|A_2 \cap A_3|} + \underset{2}{|A_1 \cap A_2 \cap A_3|}$$

$= 329$ numbers divisible by 5, 11 or 13,

thus $1001 - 329 = 672$ numbers divisible by neither

Note: Full credit if only the 329 was computed.

5) For a positive integer n we have that $2\binom{2n-1}{n} = \binom{2n}{n}$.
Give a combinatorial (i.e. describe in text a selection problem in two different ways, corresponding to the two sides of the equation), proof for this identity. No points will be given for an algebraic proof (such as the one given on the bottom) or an induction proof.

We want to select n-elt. sets from $2n$ choices. There are $\binom{2n}{n}$ such sets.

A set might not contain the largest element $2n$ then there are $2n-1$ elts to choose from, i.e. $\binom{2n-1}{n}$ choices.
Or it contains the largest elt, then the remaining $n-1$ elts are chosen from $2n-1$ smaller choices:
$\binom{2n-1}{n-1} = \binom{2n-1}{2n-1-(n-1)} = \binom{2n-1}{n}$ choices

So in total $\binom{2n-1}{n} + \binom{2n-1}{n} = 2\binom{2n-1}{n}$ choices.

Both counts must be equal, proving the statement

Algebraic Proof for which no credit would be given: $2\binom{2n-1}{n} = 2\frac{(2n-1)!}{n!(n-1)!} = \frac{2n(2n-1)!}{n!(n-1)!n} = \frac{2n!}{n!n!} = \binom{2n}{n}$.