$f = O(g)$ if $|f(x)| \le c \cdot |g(x)|$ if $x \ge N$

Often: Consider $\boxed{\lim_{x \to \infty} \frac{f(x)}{g(x)}} = L$. Then

⇔ calculate L'Hospital's Rule

$L = 0 \Rightarrow f(x) = O(g(x))$

$L = \infty \Rightarrow g(x) = O(f(x))$

L finite $\Rightarrow$ both

Ex:

$f(x) = x^2 + 1$

$g(x) = x^3 + 3x^2$

$$\lim_{x \to \infty} \frac{x^2+1}{x^3+3x^2} \overset{\text{L'Hospital}}{=} \lim_{x \to \infty} \frac{2x}{3x^2+6x} = \lim_{x \to \infty} \frac{2}{6x} = 0$$

$\Rightarrow f(x) = O(g(x))$

Ex: If $h(x) = O(f(x))$, $f(x) = O(g(x))$, then $f(x) + h(x) = O(g(x))$

h is "noise"

Ex: $f(x) = x \cdot \lg(x)$, $g(x) = x^{1.1}$

$$\lim_{x\to\infty} \frac{x \cdot \lg(x)}{x^{1.1}} \underset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{\lg(x) + x \cdot \frac{1}{x}}{1.1 \cdot x^{0.1}} \underset{\text{Limit Rule}}{=} \lim_{x\to\infty} \frac{\lg(x)}{1.1 \cdot x^{0.1}} + \lim_{x\to\infty} \frac{1}{1.1 \cdot x^{0.1}} = 0$$

$$\frac{1}{x^{0.1}} = \frac{1}{\sqrt[10]{x}}$$

$$\underset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1/x}{1.1 \cdot x^{-0.9}} = \frac{1}{1.1} \cdot \lim_{x\to\infty} \frac{1}{x^{0.1}} = 0$$

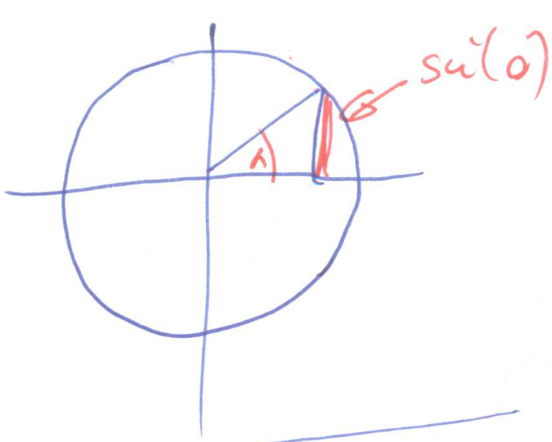
$$x^{0.1} = x^{-0.1}$$

$$\Rightarrow x \cdot \lg(x) = O(x^{1.1})$$

Ex: $f(x) = \exp(x)$, $g(x) = x^7$

$$\lim_{x\to\infty} \frac{\exp(x)}{x^7} \underset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{\exp(x)}{7 \cdot x^6} \underset{\text{L'Hospital}}{=} \cdots = \lim_{x\to\infty} \frac{\exp(x)}{\underbrace{7 \cdot 6 \cdot 5 \cdot 4 \cdots}_{=7!}} = \infty$$

$$\Rightarrow x^7 = O(\exp(x))$$



Taylor Series.

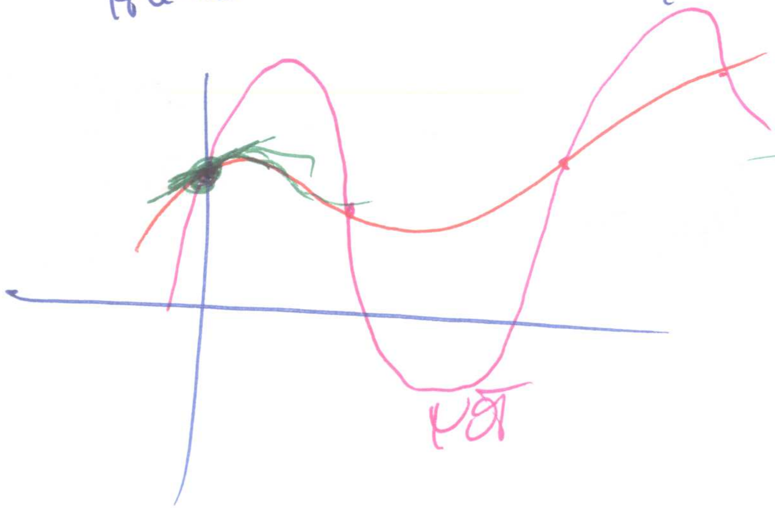
Idea: Replace/Approximate complicated functions by polynomial

Reminder: If $|x|<1$ then $\boxed{1+x+x^2+x^3+\cdots}$ "infinite polynomial"

$$= \sum_{i=0}^{\infty} x^i = \boxed{\frac{1}{1-x}} \leftarrow \text{Not polynomial}$$

§1: Taylor Polynomials

Have function $f(x)$, differentiable



Polynomial:

$$p(x) = \boxed{a_0} + \boxed{a_1} x + \boxed{a_2} x^2 + \cdots + \boxed{a_n} x^n = \sum_{i=0}^{n} a_i x^i$$

Want: good approximation.

Idea:

$p(0) = f(0)$

$p'(0) = f'(0)$

$p''(0) = f''(0)$

$p'''(0) = f'''(0)$

$$p(x) = a_0 + a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$

$p(0) = a_0$

$$p'(x) = 1 \cdot a_1 + 2a_2 x^1 + 3a_3 x^2 + \dots + \overset{4a_4 x^3}{} + n \cdot a_n x^{n-1}$$

$p'(0) = a_1$

$$p''(x) = 1 \cdot 2 \cdot a_2 + 2 \cdot 3 \cdot a_3 x + 3 \cdot 4 \cdot a_4 \cdot x^2 + \dots + (n-1) \cdot n \cdot a_n \cdot x^{n-2}$$

$p''(0) = 1 \cdot 2 \cdot a_2$

$\vdots$

$$p^{(k)} = 1 \cdot 2 \cdot 3 \cdot \dots \cdot k \cdot a_k + 2 \cdot 3 \cdots (k+1) \cdot x + \dots$$

$$p^{(k)}(0) = \underbrace{1 \cdot 2 \cdot 3 \cdots k}_{k!} \cdot a_k \overset{\text{Want}}{=} f^{(k)}(0)$$

"k-factorial"

Taylor poly formula:

$$\boxed{a_k = \frac{f^{(k)}(0)}{k!}}$$

Taylor poly of degree n for f:

$$\boxed{\sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} \cdot x^i}$$