L'Hospital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{such that} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \begin{cases} 0 \\ \pm\infty \end{cases}$$

$a = \infty$

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

ex:

$$\lim_{x\to\infty} \frac{5x^2+2x+1}{7x^2-x-5} = \lim_{x\to\infty} \frac{10x+2}{14x-1} = \lim_{x\to\infty} \frac{10}{14} = \frac{10}{14} = \frac{5}{7}$$

ex:

$$\lim_{x\to\infty} \frac{10x+1}{x^2+1} = \lim_{x\to\infty} \frac{10}{2x} = 0$$

$$\lim_{x\to\infty} \frac{x^2+1}{10x+1} = \lim_{x\to\infty} \frac{2x}{10} = \infty$$

General rule

$$\operatorname{limit}_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } \deg g > \deg f \\ \pm\infty & \text{if } \deg(g) < \deg(f) \\ \text{Quotient of leading terms} & \text{if same degree} \end{cases}$$

If both polynomials

$$\lim_{x\to\infty} \frac{\log(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{\exp(x)}{x^2} = \lim_{x\to\infty} \frac{\exp(x)}{2x} \underset{\text{again}}{=} \lim_{x\to\infty} \frac{\exp(x)}{2} = \infty$$

Order of Growth

Given an algorithm, input size $\cancel{x}$ n 1

Calculate cost function $f(x)$ n

The number of operations for input of size x

Ex: Sort "Bubble Sort"

3 5 1 2

3 1 5 2

3 1 2 5

1 3 2 5

1 2 3 5

Compare every pair of numbers

$\overset{n}{x}$ pairs

$\frac{\overset{n}{x}(\overset{n}{x}+1)}{2}$ comparisons

$\frac{1}{2}\overset{n}{x}^2 + \frac{1}{2}\overset{n}{x}$

Other algorithms $\overset{n}{x} \cdot \log(\overset{n}{x})$

Algorithm with cost f is better than alg. with cost g if then $f(x) < g(x)$ for large x

Extra conditions:

- up to constants
- "ignore noise"

Def: $f, g : \mathbb{R} \to \mathbb{R}$. Then f is of the order of g (write:

"f is not worse than g"

factor

$f = O(g)$ $\Leftrightarrow$ $f(x) = O(g(x))$) if there exists $c \in \mathbb{R}$, N

big Oh

from where on

such that $|f(x)| \leq c \, |g(x)|$ for all $x \geq N$

constant factors

large input

ex: $f(x) = x^2 + 5x$, $g(x) = x^2$

$N = 5$, $x \geq 5$

factor c

Then $x^2 + 5x = O(x^2)$, as $|x^2 + 5x| \leq x^2 + x^2 = 2 \cdot x^2$

Theorem: $f, g : \mathbb{R} \to \mathbb{R}$ s.t. $L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists. Then

a) If $L = 0$ then $f(x) = O(g(x))$

b) If $L = \infty$ then $g(x) = O(f(x))$

c) If L is finite: then $f = O(g)$ and $g = O(1)$

Midterm solution (for the class records)

1) a) $f(x) = \exp(x^2+3x)$

$$\frac{d}{dx} f(x) \underset{\text{chain rule}}{=} \exp(x^2+3x)\,(2x+3)$$

b) $x^3 \cdot \mathrm{bla}(x) \xrightarrow[\text{product rule}]{\ '\ } 3x^2 \cdot \mathrm{bla}(x) + x^3 \cdot \frac{e^x - 1}{x} = 3x^2 \cdot \mathrm{bla}(x) + x^2 \cdot (e^x - 1)$

$= x^2\left(3 \cdot \mathrm{bla}(x) + e^x - 1\right)$

c) $\frac{\cos(x)}{\sin(x)} \xrightarrow[\text{Quotient Rule}]{\ '\ } \frac{\overset{-\sin(x)}{\cancel{\cos'(x)}} \cdot \sin(x) - \cos(x) \cdot \overset{\cos(x)}{\cancel{\sin'(x)}}}{\sin^2(x)} = -\frac{\overbrace{\sin^2(x) + \cos^2(x)}^{=1}}{\sin^2(x)}$

$= -\frac{1}{\sin^2(x)}$

$$\sum_{i=0}^{n} a \cdot r^i = a\,\frac{1 - r^{n+1}}{1 - r}$$

2) $\sum_{i=10}^{50} \frac{3}{7^i} = \sum_{i=0}^{50} \frac{3}{7^i} - \sum_{i=0}^{9} \frac{3}{7^i}$ $\quad a = 3,\ r = \frac{1}{7}$

$$= 3 \cdot \frac{1 - \left(\frac{1}{7}\right)^{51}}{1 - 1/7} - 3 \cdot \frac{1 - \left(1/7\right)^{10}}{1 - 1/7}$$

3) $a_i = \dfrac{6i-1}{3i+5}$ $\lim_{i\to\infty} a_i = L = 2$

Given ε

Find N: $|a_i - L| \le \varepsilon$ if $i \ge N$;

$$\varepsilon \ge |a_n - L| = \left|\frac{6n-1}{3n+5} - 2\right| = \left|\frac{\cancel{6n}-1-\cancel{6n}-10}{3n+5}\right| = \frac{11}{3n+5}$$

Solve for n:

$$3n+5 \ge \frac{11}{\varepsilon} \Rightarrow n \ge \boxed{\frac{1}{3}\left(\frac{11}{\varepsilon}-5\right)} = N$$