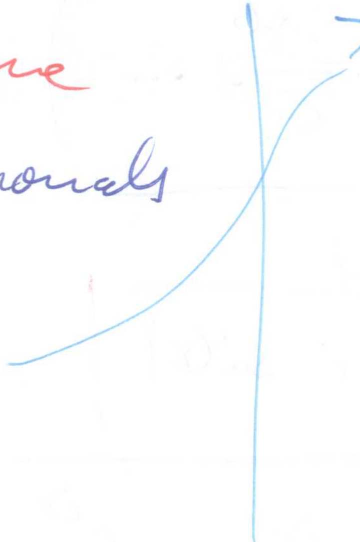
Taylor Series for $f(x)$ (around $a=0$)

$$t(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} \cdot x^{i}$$

<u>Then:</u>
- There ex. b: Series converges for $-b \le x \le b$

  Often: $b = \infty$
- If $-b \le x \le b$: Then $t(x) = f(x)$
- $t(x)$ is the <u>only</u> one

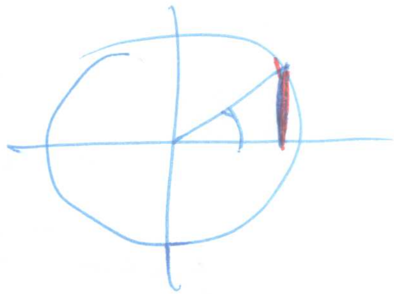
<u>Uses</u>
- Treat fcts as if polynomials
- Finding Taylor pol.
- Define (new) functions



Ex: $\exp(x) = e^{x}$

Only def for $x \in \mathbb{Q}$

$\sin(x)$



$$\exp(x) = \sum_{i=0}^{\infty} \frac{1}{i!} x^{i}$$

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^{i}}{(2i+1)!} x^{2i+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Ex: Complex Numbers

$i = \sqrt{-1}$

$$\boxed{e^{ix}} = \exp(ix) = \sum_{j=0}^{\infty} \frac{1}{j!} \underbrace{(ix)^j}_{i^j x^j}$$

$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\cos(x)} + i \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}_{\sin(x)}$$

$$= \boxed{\cos(x) + i \sin(x)}$$

Consequences: $1 + e^{i\pi} = 0$

$$\boxed{\begin{aligned} \sin(x) &= \mathrm{Im}\left(e^{ix}\right) = \frac{e^{ix} - e^{-ix}}{2i} \\ \cos(x) &= \mathrm{Re}\left(e^{ix}\right) = \frac{e^{ix} + e^{-ix}}{2} \end{aligned}}$$

Operations for Taylor Series:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \quad , \quad g(x) = \sum_{i=0}^{\infty} b_i x^i$$

Then

- $f(x)+g(x) = (f+g)(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$
- $f(x) \cdot g(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j \cdot b_{i-j} \right) x^i$
- $f'(x) = \sum_{i=0}^{\infty} i \cdot a_i x^{i-1} = \sum_{i=1}^{\infty} i \cdot a_i x^{(i-1)}$

Ex: $\exp(x) = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$

$$\frac{d}{dx} \exp(x) = \sum_{i=0}^{\infty} \frac{1}{i!} i \cdot x^{i-1}$$

$$= \sum_{i=1}^{\infty} \frac{1}{\cancel{i!}_{(i-1)!}} \cancel{i} \cdot x^{(i-1)}$$

$$\underset{j=i-1}{=} \sum_{j=0}^{\infty} \frac{1}{j!} x^j = \exp(x)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\frac{d}{dx}\sin(x) = \underbrace{1 - \frac{\cancel{3}x^2}{\underset{2!}{\cancel{3!}}} + \frac{\cancel{5}x^4}{\underset{4!}{\cancel{5!}}} - \frac{\cancel{7}\cdot x^6}{\underset{6!}{\cancel{7!}}} + \frac{\cancel{9}x^8}{\underset{8!}{\cancel{9!}}} - \dots}_{\cos(x)}$$

Ex: Taylor series for $\log(1+x)$ $\qquad$ Use $\frac{d}{dx}\log(1+x) = \frac{1}{1+x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{i=0}^{\infty} x^i \quad \text{(Geom Series)}$$

$$\uparrow -x$$

$$\frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i x^i \xrightarrow{\text{Undo Derivative}} \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \log(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} x^i}$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

Ex: Fibonacci Numbers

"generating functions"

$f_0=0,\ f_1=1,\ f_{n+2}=f_{n+1}+f_n$

$$F(x)=\sum_{i=0}^{\infty} f_i x^i = x+x^2+2x^3+3x^4+5x^5+8x^6+\dots$$

$$\boxed{xF(x)+x^2F(x)=F(x)-x}$$

Solve for $F(x) = \dfrac{x}{1-x-x^2} = x\cdot\left(\dfrac{a}{x-\alpha}+\dfrac{b}{x-\beta}\right)$

$$\alpha=\frac{-1+\sqrt{5}}{2},\quad \beta=\frac{-1-\sqrt{5}}{2}$$

$$a=-1/\sqrt{5},\quad b=\frac{1}{\sqrt{5}}$$

$$\frac{a}{x-\alpha}=\sum_{i=0}^{\infty}\frac{-a}{\alpha^{i+1}}x^i$$

Power Series.

$$F(x)=\sum_{i=1}^{\infty}\boxed{\left(\frac{-a}{\alpha^i}+\frac{-b}{\beta^i}\right)}x^i$$

Formula for Fibonacci Numbers