Ex: $f(x) = \sin(x)$ $\quad f'(x) = \cos(x)$, $f'' = -\sin(x)$, $f'''(x) = -\cos(x)$

Value at 0: $\quad 0 \quad\quad 1 \quad\quad 0 \quad\quad -1$

Taylor Pol $0 + \frac{x}{1} - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{9!} - \dots$

$$\boxed{\sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i}$$

General: The Taylor poly of degree n around a

is $\boxed{\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i}$

Theorem: $f : \mathbb{R} \to \mathbb{R}$, n-times diff'ble.

Consider interval around a of length 2ℓ

$a-\ell \dots a+\ell$

$p_n(x)$ is the degree n Taylor Polynomial.

Then $|f(x) - p_n(x)| \le \frac{M}{(n+1)!} \underbrace{|x-a|^{n+1}}_{\le \ell^{n+1}}$ $\boxed{\le \frac{M}{(n+1)!} \ell^{n+1}}$

M is upper bound for $|f^{(n+1)}(x)|$ on the interval

Ex : Approximate $\sin(x)$ on interval $-2 \ldots 2$, $a=0$, $\ell=2$

$|x-a| \leq 2$

For M: $f'(x)=\cos(x)$, $f''(x)=-\sin(x)$, $f'''(x)=-\cos(x)$

$f^{(IV)}(x)=\sin(x)$

$M=1$

Could have chosen $M=10$

Degree 3 Taylor Pol : $x-\frac{x^3}{6}=p_3(x)$

Also the degree 4 Taylor pol.:

$\text{error} \leq \frac{1}{4!}2^4 \sim 0.66$

Degree 8 Taylor Pol. $x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}$: $\text{error} \leq \frac{1}{9!}2^9$

$\sim \frac{512}{360000} \sim \frac{2}{1000}$

0.002

What if we want error $\leq 10^{-6}$

$10^{-6} \geq \frac{1}{(n+1)!}2^{n+1} \rightsquigarrow$ Try out values for n

Interval $-5 \ldots 5$

$a = 0,\ l = 5$

Ex: $f(x) = \exp(x)$

$\Rightarrow f'(x) = f''(x) = \ldots = \exp(x)$

M: Bound for $|\exp(x)|$ on $[-5 \ldots 5]$

Maximum is $\exp(5) = 2.718^5$

Use $M = 500$ $\Longleftrightarrow$ $3^5 = 243$

Taylor Series "Taylor Polynomials of degree infinity"

"Power Series"

Def: The Taylor series for $f(x)$ around $a = 0$ is

$$t(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} x^i$$

Ex: $f(x) = \exp(x)$: $\sum_{i=0}^{\infty} \frac{1}{i!} x^i$