Period Doubling Route When a parameter $\mu$ on which a dynamical system depends is varied, a periodic orbit $\Gamma(\mu)$, with period $T(\mu)$, can lose stability at a period doubling bifurcation. The typical scenario is the following: The periodic orbit has a real Floquet multiplier $m(\mu)$ such that, e.g., $1 > m(\mu) > -1$ for $\mu < \mu_0$, $m(\mu_0) = -1$, i.e. the Floquet multiplier is on the unit circle, and $m(\mu) < -1$ for $\mu > \mu_0$.\footnote{The scenario can, of course, also happen with $1 > m(\mu) > -1$ for $\mu > \mu_0$ and $m(\mu) < -1$ for $\mu < \mu_0$.} Thus, if all other Floquet multipliers are inside the unit circle at $\mu = \mu_0$, the periodic orbit is stable for $\mu < \mu_0$ and unstable for $\mu > \mu_0$. Then, generically, a unique periodic orbit $\Gamma_{pd}(\mu)$, with approximately doubled period $T_{pd}(\mu) \approx 2T(\mu)$,\footnote{More precisely: $\lim_{\mu \to \mu_0} T_{pd}(\mu) = T(\mu_0)$} bifurcates from $\Gamma(\mu)$.

As for the Andronov Hopf bifurcation, the period doubling bifurcation can be supercritical or subcritical. In the supercritical case $\Gamma_{pd}(\mu)$ exists only for $\mu > \mu_0$, in the region where $\Gamma(\mu)$ is unstable, and in the subcritical case it exists only in the region $\mu < \mu_0$ where $\Gamma(\mu)$ is stable. In both cases $\Gamma_{pd}(\mu)$ and $\Gamma(\mu)$ have opposite stabilities, i.e. $\Gamma_{pd}(\mu)$ is stable in the supercritical case and unstable in the subcritical case.

The period doubling route to chaos is a sequence of supercritical period doubling bifurcations that occur at a strictly monotonic sequence of parameter values $\mu_n$, $n = 1, 2, 3, \ldots$, with $\mu_n \to \mu_\infty$ for $n \to \infty$ and $|\mu_\infty| < \infty$. At $\mu = \mu_n$ a periodic orbit $\Gamma_n(\mu)$ undergoes a supercritical period doubling bifurcation that leads to the birth of a new stable periodic orbit $\Gamma_{n+1}(\mu)$ with doubled period, and the periods $T_n(\mu_n)$ go to $\infty$ when $\mu_n$ approaches $\mu_\infty$. Thus, when $\mu$ is varied towards $\mu_\infty$, one observes attraction to periodic orbits with larger and larger periods. Beyond the period doubling accumulation point $\mu_\infty$ the system is chaotic, and the attractor contains a countable infinity of unstable periodic orbits.

The period doubling route to chaos has been found in many model systems and experiments. It has a number of interesting universal features, in particular concerning the scaling relations along the sequence $\mu_n$. The prototype system for this scenario is the logistic map that was studied in detail by Feigenbaum in 1976.

Intermittency Route The most basic bifurcation in dynamical systems is the so called saddle node bifurcation. The simplest example is provided by $\dot{x} = x^2 + \mu$. Here for $\mu < 0$ there is a pair of fixed points $\pm \sqrt{-\mu}$ with $-\sqrt{-\mu}$ stable and $+\sqrt{-\mu}$ unstable. At $\mu = 0$ the two fixed points coalesce and for $\mu > 0$ they have disappeared.

The saddle node bifurcation can occur in any dimension and can involve fixed points as well as periodic orbits. The general bifurcation scenario is as in the example above: for, say, $\mu < \mu_0$ a (periodic or fixed point) sink and a (periodic or fixed point) saddle coexist, for $\mu = \mu_0$ they coalesce and form a “semistable” fixed point or periodic orbit, and for $\mu > \mu_0$ they have disappeared. In any case there is a “regular” (in contrast to “chaotic”) attractor for $\mu < \mu_0$ that has disappeared for $\mu > \mu_0$.

Now assume that a dynamical system undergoes a saddle node bifurcation at some value $\mu_0$, and that for $\mu < \mu_0$ the stable fixed point or periodic orbit of the stable/unstable pair is a global attractor. Since for $\mu > \mu_0$ this attractor has disappeared, trajectories must eventually leave a local region, $R$, where it existed. If a global reinjection mechanism exists that brings the trajectory back to $R$, this region is visited infinitely often. And, if $\mu - \mu_0$ is small, at each of these visits the system
spends a “long” time in $R$ (one can show that this time is of the order $O(1/\sqrt{\mu - \mu_0})$) before it leaves $R$ again. During this time the trajectory behaves like the formerly existing regular attractor, i.e. we find “nearly” periodic behavior if the attractor was a periodic orbit, or a slow drift if it was a fixed point. Thus the global behavior is characterized by so called “laminar” (regular) phases interrupted by excursions which typically manifest themselves as “chaotic bursts”. As smaller is $\mu - \mu_0$ as longer are the laminar phases. These phases are referred to as intermittency (intermittent regular dynamics). When $\mu - \mu_0$ is getting larger, the laminar phases get shorter, and the system behaves more and more chaotic. Intermittency occurs in many model systems and has also been identified in a number of experiments.

1 Period Doubling in the Rössler System

The Rössler system is the following 3d system:

\[
\begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= b + z(x - c),
\end{align*}
\]

with $b = 2$, $c = 4$ and varying $a$. This system undergoes a period doubling route to chaos. Show three dimensional solution curves displaying three successively period doubled periodic orbits for $a < 0.4$ and their $x, y, z$ time series. Also show the attractor for $a = 0.4$ in a three dimensional plot and the associated time series. The attractor for $a = 0.4$ is referred to in the literature as “Rössler band”. Then increase $a$ until you observe that the band has turned into a “funnel” (referred to as “Rössler funnel”). Show also this funnel and its time series.

**Hint:** Use a generic initial condition and discard an initial time range ($0 \leq t \leq 100$ will be sufficient) so that transients have died out. In the chaotic regime the remaining time range should be sufficiently large so that the structure of the attractor becomes apparent.

2 Period Doubling and Intermittency in the Lorenz System

Consider the Lorenz system

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= r x - y - xz, \\
\dot{z} &= xy - b z,
\end{align*}
\]

with $\sigma = 10$, $b = 8/3$, and $r$ in the range $145 \leq r \leq 166$. At the two ends of this $r$–interval the system is chaotic, but in between you should find periodic dynamics (a so called “periodic window”). Identify the transition to chaos at each of the two ends as period doubling sequence or intermittency route. Support your conclusion through two–dimensional phase portraits and the associated time series. Concerning the phase portraits, you may plot a trajectory in one of the three coordinate planes. Take again care that you have discarded transients.

**Note:** If you prefer to study the appearance of a period doubling or intermittency route to chaos in another system, possibly chosen from the text, you are welcome to do so.