Phase Oscillators and Synchronization

Synchronization in mechanical systems has been observed in the 17th century by Huygens, who noticed that two clocks attached to the same wall tend to synchronize their beats due to the interaction induced by the wall. In animal populations, synchronization is a very common phenomenon. The most striking example is probably the synchronous flashing of fireflies. Mathematically, synchronization models are described as coupled oscillators. The simplest approach is to assume that a single oscillator is described by an angular variable, called the phase, $\theta$, that describes the position on a (strongly attracting) limit cycle in the phase space of the oscillator’s dynamical system (a mapping from $S^1$ to the closed limit cycle curve in phase space). The simplest way to describe a single oscillation is then as $\dot{\theta} = \omega$, where $T = 2\pi/\omega$ is the period of the oscillation. A relation between such simplified phase oscillator models and limit cycles in some phase space has been established by Kuramoto in the 1970s.

Within the phase oscillator description, the coupling of two identical oscillators is described by a system of the form

$$
\begin{align*}
\dot{\theta}_1 &= \omega + K_1 H(\theta_2 - \theta_1) \\
\dot{\theta}_2 &= \omega + K_2 H(\theta_1 - \theta_2),
\end{align*}
$$

where $H$ is a $2\pi$-periodic function and $K_1, K_2$ are coupling constants. Introducing the phase difference $\phi = \theta_1 - \theta_2$, one gets a closed ODE for $\phi$,

$$
\dot{\phi} = K_1 H(-\phi) - K_2 H(\phi) \equiv G(\phi).
$$

Synchronization can then be described by the zeros of $G(\phi)$. If $G$ has no zeros (Equilibrium Points of the $\phi$-system), the two oscillators don’t synchronize and the joint motion is quasiperiodic. If $G$ has zeros, there is (generically) an even number, $2m$, of zeros and $m$ of them are asymptotically stable and the others are unstable. On $S^1$, stable and unstable EPs are located adjacent to each other. When $H$ depends on parameters, pairs of stable/unstable EPs can be created or destroyed in saddle node bifurcation at critical values of the parameters. The stable EPs correspond to the possible synchronized states of the two phase oscillators. (See Section 8.6 in Strogatz’s book.)

Japanese Tree Frogs

An isolated male Japanese tree frog will call nearly periodically. When two frogs are close to each other (say 50 cm apart), they can hear each other calling and tend to adjust their croak rhythms so that they call in alternation, half a cycle apart - which is referred to as antiphase synchronization.

So what happens when three frogs interact? This situation frustrates them; there’s no way, all three frogs can get half a cycle away from everyone. Aihara et al. [1] found experimentally that in this case, the three frogs settle into one of two distinctive patterns (and they occasionally seem to switch between them, probably due to noise in the environment). One stable pattern involves a pair of frogs calling in unison, with the third frog calling approximately half a cycle out of phase from both of them. The other stable pattern has the three frogs maximally out of sync, with each calling one-third of a cycle apart from the other two.
Aihara et al. explored a coupled oscillator model of these phenomena, the essence of which is contained in the system (1) for two frogs, with $K_1 = K_2 = 1$, and three frogs,

\[
\begin{align*}
\dot{\theta}_1 &= \omega + H(\theta_2 - \theta_1) + H(\theta_3 - \theta_1) \\
\dot{\theta}_2 &= \omega + H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2), \\
\dot{\theta}_3 &= \omega + H(\theta_1 - \theta_3) + H(\theta_2 - \theta_3).
\end{align*}
\]

Here $\theta_i$ denotes the phase of the calling rhythm of frog $i$ and the $2\pi$-periodic function $H$ quantifies the interaction between any two of them. The croaking of the frog can be thought of occurring when the phase is a multiple of $2\pi$. For simplicity it is assumed that the frogs are identical (same $\omega$) and are identically coupled to each other. As a further simplifying assumption assume that $H(x)$ is odd, $H(-x) = -H(x)$.

**Project:**

(a) Rewrite the systems for both two and three frogs in terms of the phase differences $\phi = \theta_1 - \theta_2$ and $\psi = \theta_2 - \theta_3$.

(b) Show that the experimental results for two frogs are consistent with the simplest possible interaction function, $H(x) = a \sin(x)$, if the sign of $a$ is chosen appropriately. But then show that this simple $H$ cannot account for the experimental three-frog results.

(c) Next consider the more complicated interaction function $H(x) = a \sin(x) + b \sin(2x)$. For the three-frog model, use a computer to plot the various phase portraits (with different dynamics) in the $(\phi, \psi)$-plane for various values of $a$ and $b$. Show that for suitable choices of $a$ and $b$, you can explain all the experimental results for two and three frogs. That is, you can find a domain in the $(a, b)$ parameter space for which the system has:

(i) a stable antiphase solution for the two-frog model

(ii) a stable phase-locked solution for the three-frog model, in which frogs 1 and 2 are in sync and approximately $\pi$ out of phase with frog 3;

(iii) a co-existing stable phase-locked solution with the three frogs one-third of a cycle apart.

(d) Show numerically that adding a small periodic component of $H$ (e.g. $\epsilon \cos(x)$) does not alter these results qualitatively.

The three-frog model studied here is more symmetrical than that considered by Aihara et al. [1]. They assumed unequal coupling strengths because in their experiments one frog was positioned midway between the other two. The frogs at either end therefore interacted less strongly with each other than with the frog in the middle. Read this paper and include a brief report of the results of these authors and how they differ from your results.

**References**