Solution of Linear Programming Problems with Matlab

Notation

- The transposition operation is denoted by a superscript T (apostrophe in Matlab),

$$[1,2,3]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = [1,2,3], \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

- Given two (row or column) vectors a and b with components $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, the notation

$$a \le b \quad \text{or} \quad a \ge b$$

  is a shorthand notation for

$$a_i \le b_i \quad \text{or} \quad a_i \ge b_i \quad \text{for all} \quad 1 \le i \le n.$$

Definition 1. Let f be a column vector of length n, b a column vector of length m, and let A be a $m \times n$–matrix.

A linear program associated with f, A, and b is the minimum problem

$$\min f^T x \tag{1}$$

or the maximum problem

$$\max f^T x \tag{2}$$

subject to the constraint

$$Ax \le b. \tag{3}$$

Note that x is a column vector of length n.

The general version of a linear program may involve inequality constraints as well as equality constraints:

Definition 2. Let f be a column vector of length n, b a column vector of length m, b_{eq} a column vector of length k, and let A and A_{eq} be $m \times n$ and $k \times n$ matrices, respectively.

A linear program associated with f, A, b, A_{eq}, b_{eq} is the minimum problem (1) or the maximum problem (2), subject to the inequality constraint (3) *and* the equality constraint

$$A_{eq} x = b_{eq}. \tag{4}$$

Example. The winemaker example led us to the following problem:

$$12x_1 + 7x_2 = \max,$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 10,000 \\ 3x_1 + 2x_2 &\leq 16,000 \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

If we define

$$f = \begin{bmatrix} 12 \\ 7 \end{bmatrix}, \quad b = \begin{bmatrix} 10,000 \\ 16,000 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

this problem can be identified with the linear programming maximum problem associated with f, A, b. Likewise it can be identified with the linear programming minimum problem associated with $-f$, A, b.

Solution of linear programming minimum problems with Matlab

Matlab provides the command *linprog* to find the minimizer (solution point) x of a linear programming minimum problem. Without equality constraint the syntax is

```
x=linprog(f,A,b)
```

If you also want to retrieve the minimal value $f_{min} = \min_x(f^T x)$, type

```
[x,fmin]=linprog(f,A,b)
```

If inequality *and* equality constraint are given, use the commands

```
x=linprog(f,A,b,Aeq,beq)
```

or

```
[x,fmin]=linprog(f,A,b,Aeq,beq)
```

Let's solve our winemaker problem:

```
>> f=[-12;-7];b=[10000;16000;0;0];A=[2 1;3 2;-1 0;0 -1];
>> [x,fopt]=linprog(f,A,b)
Optimization terminated successfully.

x =

  1.0e+003 *

   3.99999999989665
```

```
   2.00000000013951


fopt =

   -6.199999999973631e+004
```

This is the answer found in the class notes. The solution point is $(4000, 2000)$, and the maximum profit is $\$6,2000$.

Practice Problems

In each of the following problems first identify vectors and matrices such that the optimization problem can be written in the form of Definitions 1 or 2. Then use the *linprog* command to solve the linear program.

Problem 1.

$$x_1 + x_2 = \max$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 29, \\ x_1 + 2x_2 &\leq 25, \\ x_1 &\geq 2, \\ x_2 &\geq 5. \end{aligned}$$

Problem 2.

$$x_1 + x_2 + x_3 + x_4 + x_5 = \max$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 100, \\ x_3 + x_4 &\leq 70, \\ x_2 + x_3 + 2x_4 + 5x_5 &\leq 250, \\ x_j &\geq 0 \ \ (1 \leq j \leq 5). \end{aligned}$$

Problem 3.

$$x_1 + x_2 + x_3 + x_4 + x_5 = \min$$

subject to

$$\begin{aligned} x_1 + x_2 &= 100, \\ x_3 + x_4 &= 70, \\ x_2 + x_3 + 2x_4 + 5x_5 &= 250, \\ x_j &\geq 0 \ \ (1 \leq j \leq 5). \end{aligned}$$