Solution of Linear Programming Problems with Matlab

Notation

- The transposition operation is denoted by a superscript $T$ (apostrophe in Matlab),

$$[1, 2, 3]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = [1, 2, 3], \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

- Given two (row or column) vectors $a$ and $b$ with components $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, the notation

$$a \leq b \quad \text{or} \quad a \geq b$$

is a shorthand notation for

$$a_i \leq b_i \quad \text{or} \quad a_i \geq b_i \quad \text{for all} \quad 1 \leq i \leq n.$$ 

Definition 1. Let $f$ be a column vector of length $n$, $b$ a column vector of length $m$, and let $A$ be a $m \times n$–matrix.

A linear program associated with $f$, $A$, and $b$ is the minimum problem

$$\min f^T x$$

or the maximum problem

$$\max f^T x$$

subject to the constraint

$$Ax \leq b.$$ 

Note that $x$ is a column vector of length $n$.

The general version of a linear program may involve inequality constraints as well as equality constraints:

Definition 2. Let $f$ be a column vector of length $n$, $b$ a column vector of length $m$, $b_{eq}$ a column vector of length $k$, and let $A$ and $A_{eq}$ be $m \times n$ and $k \times n$ matrices, respectively.

A linear program associated with $f$, $A$, $b$, $A_{eq}$, $b_{eq}$ is the minimum problem (1) or the maximum problem (2), subject to the inequality constraint (3) and the equality constraint

$$A_{eq}x = b_{eq}.$$ 

Example. The winemaker example led us to the following problem:

$$12x_1 + 7x_2 = \max,$$
subject to

\[
\begin{align*}
2x_1 + x_2 & \leq 10,000 \\
3x_1 + 2x_2 & \leq 16,000 \\
x_1 & \geq 0, \\
x_2 & \geq 0.
\end{align*}
\]

If we define

\[
f = \begin{bmatrix} 12 \\ 7 \end{bmatrix}, \quad b = \begin{bmatrix} 10,000 \\ 16,000 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},
\]

this problem can be identified with the linear programming maximum problem associated with \( f, A, b \). Likewise it can be identified with the linear programming minimum problem associated with \(-f, A, b\).

**Solution of linear programming minimum problems with Matlab**

Matlab provides the command `linprog` to find the minimizer (solution point) \( x \) of a linear programming minimum problem. Without equality constraint the syntax is

\[
x = \text{linprog}(f, A, b)
\]

If you also want to retrieve the minimal value \( f_{\text{min}} = \min_x (f^T x) \), type

\[
[x, f_{\text{min}}] = \text{linprog}(f, A, b)
\]

If inequality and equality constraint are given, use the commands

\[
x = \text{linprog}(f, A, b, A_{\text{eq}}, b_{\text{eq}})
\]

or

\[
[x, f_{\text{min}}] = \text{linprog}(f, A, b, A_{\text{eq}}, b_{\text{eq}})
\]

Let’s solve our winemaker problem:

\[
\begin{align*}
& f = [-12; -7]; b = [10000; 16000; 0; 0]; A = [2 1; 3 2; -1 0; 0 -1]; \\
& [x, f_{\text{opt}}] = \text{linprog}(f, A, b)
\end{align*}
\]

Optimization terminated successfully.

\[
x =
\begin{align*}
1.0e+003 & \quad \ast \\
3.99999999989665 & \quad \ast
\end{align*}
\]
2.000000000013951

fopt =

\[-6.199999999973631e+004\]

This is the answer found in the class notes. The solution point is \((4000, 2000)\), and the maximum profit is $6,2000.

**Practice Problems**

In each of the following problems first identify vectors and matrices such that the optimization problem can be written in the form of Definitions 1 or 2. Then use the *linprog* command to solve the linear program.

**Problem 1.**

\[
x_1 + x_2 = \text{max}
\]

subject to

\[
\begin{align*}
2x_1 + x_2 & \leq 29, \\
x_1 + 2x_2 & \leq 25, \\
x_1 & \geq 2, \\
x_2 & \geq 5.
\end{align*}
\]

**Problem 2.**

\[
x_1 + x_2 + x_3 + x_4 + x_5 = \text{max}
\]

subject to

\[
\begin{align*}
x_1 + x_2 & \leq 100, \\
x_3 + x_4 & \leq 70, \\
x_2 + x_3 + 2x_4 + 5x_5 & \leq 250, \\
x_j & \geq 0 \ (1 \leq j \leq 5).
\end{align*}
\]

**Problem 3.**

\[
x_1 + x_2 + x_3 + x_4 + x_5 = \text{min}
\]

subject to

\[
\begin{align*}
x_1 + x_2 & = 100, \\
x_3 + x_4 & = 70, \\
x_2 + x_3 + 2x_4 + 5x_5 & = 250, \\
x_j & \geq 0 \ (1 \leq j \leq 5).
\end{align*}
\]