Exam 1 (each problem is worth 100 points)

1: Find the explicit solution of the initial value problem and state the interval of existence.

\[ \frac{dy}{dx} = \frac{x}{y(x^2 - 1)}, \quad y(0) = 1 \]

The equation is separable:

\[ y \, dy = \frac{x}{x^2 - 1} \, dx \]

Integrate this:

\[ \int y \, dy = \int \frac{x}{x^2 - 1} \, dx \]

\[ \Rightarrow \quad \frac{1}{2} y^2 = \frac{1}{2} \ln |x^2 - 1| + C \]

Invoke IC: \( \frac{1}{2} C = C \Rightarrow y^2 = \ln |x^2 - 1| + 1 \)

\[ \Rightarrow \quad y(x) = \sqrt{\ln(1-x^2) + 1} \quad \text{is solution of IVP} \quad (x^2 < 1) \]

Find I0E: \( \ln (1-x^2) > -1 \)

\[ \Rightarrow \quad 1-x^2 > e^{-1} \Rightarrow x^2 < 1-1/e \]

\[ \Rightarrow \quad I_{0E}: \quad -\sqrt{1-1/e} < x < \sqrt{1-1/e} \]
2: (a) Find the general solution of the following equation:
\[
\frac{dy}{dt} + y \cos t = \cos t
\]

Use integrating factor: \( u(t) = e^{\int \cos t \, dt} = e^{\sin t} \)

Multiply equation by \( e^{\sin t} \):
\[
e^{\sin t} \frac{dy}{dt} + ye^{\sin t} \cos t = e^{\sin t} \cos t = \frac{d}{dt}(e^{\sin t} y) = e^{\sin t} \cos t
\]

Integrate:
\[
e^{\sin t} y = \int e^{\sin t} \cos t \, dt = e^{\sin t} + C
\]

\[
\Rightarrow y(t) = 1 + Ce^{-\sin t} \quad \text{(general solution)}
\]

(b) Find the solution of the following initial value problem:
\[
\frac{dy}{dt} = -2ty + 3t^2 e^{-t^2}, \quad y(0) = 1
\]

Use variation of parameter: \( y_h(t) = e^{\int (-2t) \, dt} = e^{-t^2} \)

Particular solution:
\[
y_p(t) = e^{-t^2} \int 3t^2 \frac{e^{-t^2}}{e^{-t^2}} \, dt = e^{-t^2} t^3
\]

General solution:
\[
y(t) = C e^{-t^2} + t^3 e^{-t^2}
\]

IC: \( y(0) = C = 1 \)

\[
\Rightarrow y(t) = e^{-t^2} (1 + t^3) \quad \text{is solution of IVP.}
\]
1.3: Find the general solution in implicit form for the equation below. *Hint:* The differential form associated with this equation is exact.

\[
\frac{dy}{dx} = \frac{5x^4 - y^2}{2xy + 3y^2}
\]

**Differential form:**

\[
\left( y^3 - 5x^4 \right) \frac{dx}{P(x,y)} + \left( 2xy + 3y^2 \right) \frac{dy}{Q(x,y)} = 0
\]

\[
P_y = 2y, \quad Q_x = 2y = P_y \implies \text{equation is exact}
\]

Set \( F(x,y) = \int P \, dx + \phi(y) \)

\[
= \int \left( y^3 - 5x^4 \right) dx + \phi(y)
\]

\[
= y^2x - x^5 + \phi(y)
\]

require \( \frac{\partial F}{\partial y} = 2xy + \phi'(y) = Q = 2xy + 3y^2 \)

\[
\implies \phi' = 3y^2 \implies \phi(y) = y^3
\]

**General implicit solution:**

\[
F(x,y) = xy^3 - x^5 + y^3 = C
\]
4: Find the general solution in implicit form for the equation below. *Hint:* The equation is homogeneous.

\[
\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}
\]

**equation is homogeneous; set \( y = xu \)**

Then \( \frac{dy}{dx} = x \frac{dv}{dx} + vu = \frac{1 + 3v^2}{2v} \)

\[
= \frac{x}{dx} \frac{dv}{dx} = \frac{1 + 3v^2}{2v} - u = \frac{1 + v^2}{2v}
\]

**separate:** \( \frac{2v}{1 + v^2} dv = \frac{dx}{x} \)

**integrate this:** \( \int \frac{2v}{1 + v^2} dv = \int \frac{dx}{x} \)

\[
\Rightarrow \ln(1 + v^2) = \ln|x| + C = \ln(e^C |x|)
\]

\[
\Rightarrow 1 + v^2 = e^C |x|
\]

\[
\Rightarrow 1 + v^2 = Dx \quad (D = \pm e^C \text{ s.t. } Dx \geq 0)
\]

**Substitute \( u = \frac{y}{x} \):**

\[
\Rightarrow 1 + \frac{y^2}{x^2} = Dx
\]

\[
\Rightarrow x^2 + y^2 = Dx^3 \quad \text{is general solution}
\]
5: Just before midday the body of an apparent homicide victim is found in a room that is kept at a constant temperature of 70°F. At 12 noon the temperature of the body is 80°F, and at 1 pm it is 75°F. Assuming that the temperature of the body at the time of death was 100°F and that it has cooled in accord with Newton's law of cooling,

\[ \frac{dT}{dt} = k(A - T), \quad A: \text{room temperature,} \quad k > 0, \]

when did the homicide occur? You may use numerical approximations from the table below in your calculation.

<table>
<thead>
<tr>
<th>ln(2)</th>
<th>ln(3)</th>
<th>ln(4)</th>
<th>ln(5)</th>
<th>ln(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.1</td>
<td>1.4</td>
<td>1.6</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Solution: \( T(t) = (T_0 - A)e^{-kt} + A, \quad A = 70°F \)

At noon \( t=0 \): \( T_0 = 80°F \)

At 1 pm \( t=1 \): \( T(1) = (80-70)e^{-k} + 70 = 75 \)

\[ e^k = \frac{80-70}{75-70} = \frac{10}{5} = 2 \]

\[ k = \ln 2 \approx 0.7/hr \]

Then \( T(t) = 10e^{-kt} + 70 \) with \( k = \ln 2 \),

and \( T(1) = 100 \) gives \( 10e^{-kt} + 70 = 100 \)

\[ e^{-kt} = 3 \quad \Rightarrow \quad -kt = \ln 3 \]

\[ t = -\frac{\ln 3}{\ln 2} \approx -\frac{1.1}{0.7} \approx -1.6 = -(1 \text{ hr } 36 \text{ min}) \]

\( \) homicide occurred at 10:24 am
Consider the autonomous equation
\[
\frac{dx}{dt} = (x + 2)(x + 1)(x - 1)(x - 2) = x^4 - 5x^2 + 4.
\]

Find all equilibrium points, classify their stability, and sketch the phase line diagram.

Use this information to sketch the solution curves in the \((t, x)\)-plane \((t \geq 0)\) for the initial conditions \(x(0) = -2, x(0) = -1.5, x(0) = 0, x(0) = 1\).

**EPs:** \(x_e = -2, -1, 1, 2\)

**Stability:**
- \(f'(x) = 4x^3 - 10x\)
- \(f'(-2) = -32 + 20 < 0 \Rightarrow -2\) is stable
- \(f'(-1) = -4 + 10 > 0 \Rightarrow -1\) is unstable
- \(f'(1) = 4 - 10 < 0 \Rightarrow 1\) is stable
- \(f'(2) = 32 - 20 > 0 \Rightarrow 2\) is unstable.

**Phase line diagram:**

![Phase line diagram](image)

**Solution curves:**

![Solution curves](image)