APPORTIONMENT

Goal: Divide objects that are the same but each person or "state" gets a different amount.

Example:

Three clubs sending reps to a student council, Drama Club with 33 members, Garden Club with 33 members, and Rodeo Club with 34 members. There are 10 seats available on the council.

How do we divide the seats?

Drama Club and Garden Club should get 33.3% of the council seats, or 3.3 seats.

Rodeo Club should get 34% of the council seats, or 3.4 seats.

Cannot divide seats, so what to do?

Try rounding: each club gets three seats.

Apportioned 9 out of 10 seats- Who gets the last seat?

Need a better apportionment method…

HAMILTON'S METHOD

Step 1:
Calculate each state's Standard Quota

$$\text{STANDARD DIVISOR} = \frac{\text{TOTAL POPULATION}}{\text{NUMBER OF SEATS}}$$

$$\text{STANDARD QUOTA} = \frac{\text{STATE POPULATION}}{\text{STANDARD DIVISOR}}$$

LOWER QUOTA – ROUND DOWN

UPPER QUOTA – ROUND UP

Step 2: Allocate the Lower Quota (i.e. Round Down)

Step 3: Give the surplus seats to the state with the largest fractional parts until no more surplus seats

Example:

Three clubs sending reps to a student council, Drama Club with 33 members, Garden Club with 33 members, and Rodeo Club with 34 members.

With 10 seats, the Standard Divisor is: 100/10 = 10

Club	Std. Quota	Lower Quota
Drama	33/10=3.3	3
Garden	33/10=3.3	3
Rodeo	34/10=3.4	3

Total seats apportioned = 9

Surplus = 1, goes to Rodeo, largest fractional part (3.4 vs. 3.3)

EXAMPLE

Banana Republic has states Apure, Barinas, Carabobo, and Dolores and 160 seats in the legislature.

State	A	B	C	D
Pop. (in millions)	3.31	2.67	1.33	0.69

Step 1: Calculate Standard Quota
Standard Divisor:
(3.31 + 2.67 + 1.33 + .69)/160 = .05

Standard Quota

State	Std. Quota	Lower Quota
Apure	3.31/.05=66.2	66
Barinas	2.67/.05=53.4	53
Carabobo	1.33/.05=26.6	26
Dolores	.69/.05=13.8	13

Step 2: Allocate Lower Quota

Step 3: Allocate Surplus
Total Allocated = 158
Surplus = 160 – 158 = 2
D gets one more seat, C gets one more seat

Total Allocation:
A = 66
B = 53
C = 27
D = 14

Example

The Scotia Metro Area Rapid Transit operates 6 bus routes (A-F) and 130 buses, apportioned based on the average number of daily passengers:

A	B	C	D	E	F
45,300	31,070	20,490	14,160	10,260	8,720

Calculate Quotas
Standard Divisor :
Sum of all passengers: 130,000
130,000/130=1000

Std Quota:

Route	Std. Quota	Lower Quota
A	45,300/1000=45.3	45
B	31,070/1000=31.07	31
C	20,490/1000=20.49	20
D	14,160/1000=14.16	14
E	10,260/1000=10.26	10
F	8,720/1000=8.72	8

Allocate Surplus
Total Allocated = 128
Surplus = 130 – 128 = 2
F gets one more, C gets one more
Total Allocation:

A = 45	D = 14
B = 31	E = 10
C = 21	F = 9

Example

A mother wishes to distribute 11 pieces of candy among 3 children based on the time each child spends studying:

Child	Bob	Peter	Ron
Time	54	243	703

Calculate Standard quotas
Standard Divisor = (54 + 243 + 703)/11 = 90.9

Child	Std Quota	Lower Quota
Bob	54/90.9=.59	0
Peter	243/90.9=2.67	2
Ron	703/90.9=7.73	7

Allocate Lower Quota

Allocate Surplus
Total Allocated = 9
Total Surplus = 11 – 9 = 2
Ron gets one more, Peter gets one more

Total Allocation:
Bob = 0
Peter = 3
Ron = 8

<u>Lowndes's Method</u>

Same as Hamilton's Method until the step of apportioning the surplus seats.

Hamilton's Method looks at *absolute* fraction.

Lowndes's Method looks at *relative* fraction.

Example: 250 seats are being apportioned among 6 states. State B has 6,936,000 people and State E has 685,000 people.

Standard Quotas: B:138.72 E: 13.70

Under Hamilton's method, an extra seat would go to B before E, but that seat would mean much more to E than B.

Use relative fractional parts to describe this mathematically .

B: 0.72/138=0.00522
E: 0.7/13=0.0538

E would get the extra seat under Lowndes's Method.

The Quota Rule

A state's apportionment should be either its upper quota or its lower quota. An apportionment method that guarantees that this will happen is said to satisfy the Quota Rule.

Reminder:

$$\text{STANDARD DIVISOR} = \frac{\text{TOTAL POPULATION}}{\text{NUMBER OF SEATS}}$$

$$= \text{NUMBER OF PEOPLE PER SEAT}$$

$$\text{STANDARD QUOTA} = \frac{\text{STATE POPULATION}}{\text{STANDARD DIVISOR}}$$

LOWER QUOTA – STD QUOTA, ROUND DOWN
UPPER QUOTA – STD QUOTA, ROUND UP

Hamilton's Method always satisfies the Quota Rule – state will receive lower quota, or 1 from surplus = upper quota.

Does Lowndes's Method satisfy the Quota Rule?

PARADOXES

Use Hamilton's Method:

House of Reps, 1882
299 Seats available
Alabama, Texas and Illinois have 35 seats

State	Std Quota	Lower Q	Apportioned
Alabama	7.646	7	8
Texas	9.64	9	9
Illinois	18.64	18	18
Total seats:		34	35

Increase number of seats to 300
Alabama, Texas and Illinois have 36 seats

State	Std Quota	Lower Q	Apportioned
Alabama	7.671	7	7
Texas	9.672	9	10
Illinois	18.702	18	19
Total seats:		34	36

Not fair to Alabama to lose a seat just because the total number of seats increased.

Called Alabama Paradox

Especially unfair for small states that might bounce between 2 and 3 representatives, depending not on that state's population, but the mathematics. After 1901: Hamilton's Method was no longer used.

Another Example:

2525 Intergalactic Federation Population figures (in billions)

Planet	Alanos	Betta	Conii	Digos	Ellisium	Total
Population	150	78	173	204	295	900

There are 50 seats available.
Standard Divisor: 18 billion

Apportionment:

Planet (2525)	Pop (in billions)	Std. Quota	Lower Quota	Surplus	Final
Alanos	150	$8.\overline{3}$	8		8
Betta	78	$4.\overline{3}$	4		4
Conii	173	$9.6\overline{1}$	9	1	10
Dugos	204	$11.\overline{3}$	11		11
Ellisium	295	$16.3\overline{8}$	16	1	17
Total	900	50.00	48	2	50

10 Years Later, population increase, 50 seats

Standard Divisor: 18.18 billion

Apportionment:

Planet (2535)	Pop (in billions)	Std. Quota	Lower Quota	Surplus	Final
Alanos	150	8.25	8		8
Betta	78	4.29	4	1	5
Conii	181	9.96	9	1	10
Dugos	204	11.22	11		11
Ellisium	296	16.28	16		16
Total	909	50.00	48	2	50

Note that Betta's population remained the same, or grew rather slowly, but Ellisium's population grew much faster. Is it fair that Ellisium lost a seat to Betta?

Population Paradox: X loses a seat to Y even though X's population grew at a higher rate than Y's.

What happens if a population increase results in more seats and a new state?
(watch the apportionment of South High)

Old Apportionment of Counselors to Two High Schools

School	Enrollment	Std. Quota	Apportion
North High	1045	10.45	10
South High	8955	89.55	90
Total	10,000	100.00	100

Add a District: 525 students, 5 counselors

School	Enrollment	Std. Quota	Apportion
North High	1045	10.42	11
South High	8955	89.34	89
New High	525	5.24	5
Total	10,525	105	105

New-States Paradox: Adding a new state and seats can affect the apportionment of other states.

Summary:

$$\text{STANDARD DIVISOR} = \frac{\text{TOTAL POPULATION}}{\text{NUMBER OF SEATS}}$$

$$= \text{NUMBER OF PEOPLE PER SEAT}$$

$$\text{STANDARD QUOTA} = \frac{\text{STATE POPULATION}}{\text{STANDARD DIVISOR}}$$

LOWER QUOTA – STD QUOTA, ROUND DOWN
UPPER QUOTA – STD QUOTA, ROUND UP

Apportionment Methods that Satisfy the Quota Rule:

- **Hamilton's (largest fraction gets surplus)**
- **Lowndes's (largest relative fraction gets surplus)**

Paradoxes:

- **Alabama (increase in # of seats)**
- **Population (increase in population)**
- **New-States (increase in states and seats)**