A DESCRIPTION OF DE RHAM COHOMOLOGY

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1. Motivation

Consider the following problem, which one might come across if taking a Calculus III course:

1. Determine if the vector field $\vec{F} = \langle -y, x \rangle$ is conservative.

In Calculus III, students are taught that one may test if a vector field is conservative by checking if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. In the case of $\vec{F} = \langle -y, x \rangle$, computing the partial derivatives, one has

$$\frac{\partial}{\partial y}(-y) = -1 \neq 1 = \frac{\partial}{\partial x}(x)$$

so by the partial derivative test, this would not be a conservative vector field.

However, if one slightly modifies the vector field by 'normalizing' we arrive at a problem that would be beyond the scope of Calculus III. Consider the following problem, which one should never come across if taking a Calculus III course:

2. Determine if the vector field below is conservative:

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

That is because this vector field passes the partial derivative test mentioned before, yet there is no potential function f for $\vec{F}$, i.e. a function f such that $\vec{F} = \vec{\nabla}(f)$.

Indeed, computing the partial derivatives yields

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{x^2+y^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

so naively applying the partial derivative test, one is led to believe that $\vec{F}$ is a conservative vector field.

However, consider that the following fact would fail to hold:

Theorem 1.1. If $\vec{F}$ is conservative, then $\oint_c \vec{F} \cdot \vec{T} ds = 0$ for all loops c.

If c is a path on the circle $x^2 + y^2 = 1$ oriented clockwise looping around exactly once, then $\oint_c \vec{F} \cdot \vec{T} ds = 2\pi \neq 0$.

What is failing is that the partial derivative test $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we applied doesn't take into account the domain on which the vector field is defined. Note that the domain of $\vec{F}$ from problem 2 is $\mathbb{R}^2 \setminus \{0\}$, which has singular cohomology groups $H^0 \cong \mathbb{Z}, H^1 \cong \mathbb{Z}$ and $H^{i>1} \cong 0$.

As it turns out, vector-field like objects can be leveraged to define a cohomology theory for spaces where one can differentiate functions between the spaces. This is made precise in defining de Rham cohomology.

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2. Background

Definition 2.1. [1] A topological n-dimensional manifold M with atlas A is called *smooth* if for any charts $\varphi_1, \varphi_2 \in A$ with $\varphi_1 : U \to B_1$ and $\varphi_2 : V \to B_2$ for open $U, V \subseteq M$ (see Figure 1), the transition map

$$(\varphi_2 |_{U \cap V}) \circ (\varphi_1 |_{U \cap V})^{-1} : B_1 \to B_2$$

is smooth (i.e. C^∞) as a map of open subsets of $\mathbb{R}^n$.

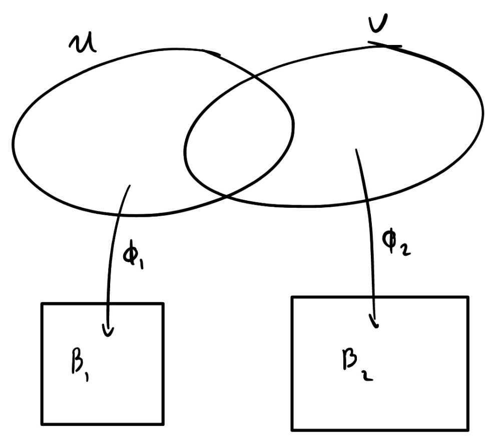

Figure 1. Two overlapping charts for a manifold M.

Definition 2.2. Let (M, A) and (N, B) be smooth manifolds with $p \in M$. A continuous map $f : M \to N$ is smooth at p if for any chart $\varphi : U \to B_1$ and $\psi : V \to B_2$ where $f(p) \in V$, $\varphi \in A$, and $\psi \in B$, then the map

$$\psi \circ f |_{U \cap f^{-1}(V)} \circ (\varphi |_{U \cap f^{-1}(V)})^{-1} : \varphi(U \cap f^{-1}(V)) \to B_2$$

is smooth at $\varphi(p)$.

The map f is *smooth* if it is smooth at all points $p \in M$.

Definition 2.3. If (M, A) and (N, B) are smooth manifolds, then a *smooth homotopy* H between a pair of maps $f, g : M \to N$ is a map $H : M \times \mathbb{R} \to N$ such that H is a homotopy between f and g and H is a smooth map viewing $M \times \mathbb{R}$ as a product of smooth manifolds.

Definition 2.4. For (M, A) a smooth manifold with $p \in M$, consider all paths $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$. The *tangent space* $T_p(M)$ is the collection of such paths modulo the equivalence relation that $\gamma_1 \sim \gamma_2$ if $(\gamma_1)'(0) = (\gamma_2)'(0)$. Note that by the linearity of the derivative, the tangent space is a real vector space.

The tangent space $T_p(M)$ can be viewed as an inner product space with the real dot product, and that if M is n-dimensional, then $T_p(M) \cong \mathbb{R}^n$ as vector spaces.

Definition 2.5. [3] A degree k *differential form* φ is a smooth assignment of a k-multilinear map $\varphi : (T_p(M))^k \to \mathbb{R}$ for each point $p \in M$.

If the smooth manifold M has local coordinates $(x_1, \dots, x_n)$ at p, then a monomial k-form φ can be written as $f(x_1, \dots, x_n) dx_{i_1} \cdots dx_{i_k}$ where each $dx_{i_j} : T_p(M) \to \mathbb{R}$ is a smooth linear functional and $f(x_1, \dots, x_n)$ is a smooth function $f : M \to \mathbb{R}$. The collection of k-forms on M is denoted $\Omega^k(M)$ which forms a vector space with 0 vector given by $0dx_I$, where I is a cardinality k indexing set.

Definition 2.6. The *exterior derivative* of a monomial k-form $\omega = f(x_1, \dots, x_n) dx_{i_1} \cdots dx_{i_k}$ is

$$d\omega = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j dx_{i_1} \cdots dx_{i_k}$$

To define the exterior derivative of a general k-form, this definition is extended linearly.

Note that the exterior derivative is a map from $\Omega^k(M) \to \Omega^{k+1}(M)$, i.e. it sends k-forms to $k+1$ forms.

Definition 2.7. A *closed form* is a k-form φ with null exterior derivative, i.e. $d\varphi = 0$.

Definition 2.8. An *exact form* is a k-form φ such that there exists a $(k-1)$ form ψ where $\varphi = d\psi$.

3. The de Rham Cohomology

Theorem 3.1. For any smooth manifold M, $(\Omega^\bullet(M), d)$ forms a cochain complex.

This follows from the fact that the exterior derivative satisfies $d^2 = 0$, i.e. $d(d\varphi) = 0$. This is exactly the statement that exact differential forms are closed (Poincare's Lemma).

Definition 3.2. [3] The *de Rham Cohomology* of a smooth manifold M is the cohomology of the cochain complex $(\Omega^\bullet(M), d)$,

$$H^i_{\text{dR}}(M) \coloneqq \frac{\ker d^i}{\text{Im} d^{i-1}}$$

Definition 3.3. The induced map for smooth $f : M \to N$ by de Rham cohomology is

$$f^i : H^i_{\text{dR}}(M) \to H^i_{\text{dR}}(N)$$

given by the pullback of i-forms along f.

Definition 3.4. Let (M, A) be a pair of smooth manifolds with inclusion $i : A \to M$. Then the cochain complex on M relative to A is

$$\Omega^*(M, A) \coloneqq Ker(i^*)$$

where $i^* : \Omega^*(M) \to \Omega^*(A)$ is the induced map on differential forms. Note that together with the exterior derivative d, the $\Omega^*(M, A)$ forms a relative cochain complex. Denote the restriction of d to the relative cochain complex as d_{rel}. Define the *relative de Rham cohomology* of (M, A) to be the cohomology of $(\Omega^*(M, A), d_{\text{rel}})$,

$$H^i_{\text{dR}}(M) \coloneqq \frac{\ker d^i_{\text{rel}}}{\text{Im} d^{i-1}_{\text{rel}}}$$

4. Two properties from class

Theorem 4.1. [3] Let M and N be smooth manifolds with $f : M \to N$ a smooth homotopy equivalence. Then $f^i : H^i_{\mathrm{dR}}(M) \to H^i_{\mathrm{dR}}(N)$ is an isomorphism for each i.

Theorem 4.2. Let A and B be submanifolds of M such that (M, A) and (M, B) are smooth pairs. If $A \cup B = M$ and $\partial A \cap \partial B = \emptyset$, then there is a long exact sequence:

$$\cdots \to H^i_{\mathrm{dR}}(M) \to H^i_{\mathrm{dR}}(A) \oplus H^i_{\mathrm{dR}}(B) \to H^i_{\mathrm{dR}}(A \cap B) \to H^{i+1}_{\mathrm{dR}}(M) \to \cdots$$

This long exact sequence is the Mayer-Vietoris sequence for de Rham cohomology.

5. Two properties not from class

Theorem 5.1. If M is a smooth manifold with boundary, then

$$H^i_{\mathrm{dR}}(M \times S^1) \cong H^i_{\mathrm{dR}}(M) \oplus H^{i-1}_{\mathrm{dR}}(M)$$

Theorem 5.2 (de Rham's Theorem)**.** [2] Let M be a smooth manifold. Then the de Rham cohomology and the singular cohomology of M with real coefficients coincide, that is, $H^i_{\mathrm{dR}}(M) \cong H^i(M; \mathbb{R})$.

Note that this statement provides an answer to Elie Cartan's conjecture that the Betti numbers of a smooth manifold could be captured by differential forms, and the de Rham Theorem connects the smooth structure of a smooth manifold to the topology of the smooth manifold.

6. Examples

The following computations proceed inductively rather than utilizing either of the theorems mentioned above.

Proposition 6.1. $H^i_{\mathrm{dR}}(S^n) \cong \mathbb{R}$ for $i = 0$, $i = n$ and $H^i_{\mathrm{dR}}(S^n) \cong 0$ otherwise.

Proof. We proceed inductively. Indeed, $H^i_{\mathrm{dR}}(X \sqcup Y) = H^i_{\mathrm{dR}}(X) \oplus H^i_{\mathrm{dR}}(Y)$, and for $M = \{p\}$ a point, $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$, and $H^i_{\mathrm{dR}}(M) \cong 0$ for $i \geq 1$.

For the base case, let $n = 1$. For S^1, we let A be $S^1 \setminus \{s\}$ where s is the south pole and B be $S^1 \setminus \{n\}$ where n is the north pole. Then $A \cap B \simeq S^0$ and $A \simeq B \simeq \mathbb{R}^0$. Considering the Mayer-Vietoris sequence in Figure 2, we have that the maps a and e are 0 and c is a matrix of 1's, so $\ker c \cong \mathbb{R}$ and $\mathrm{Im} c \cong \mathbb{R}$. By exactness, $\ker b \cong \mathrm{Im} a \cong 0$ and $\mathrm{Im} b \cong \ker c \cong \mathbb{R}$ making $H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}$. Also by exactness, $\ker d \cong \mathrm{Im} c \cong \mathbb{R}$ and $\mathrm{Im} d \cong \ker e \cong H^1_{\mathrm{dR}}(S^1)$ making $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$. Note that all the higher de Rham cohomology groups are trivial for A, B, and $A \cap B$, hence $H^i_{\mathrm{dR}}(S^1) \cong 0$ for $i > 1$.

$$H^{-1}_{\mathrm{dR}}(S^0) \xrightarrow{a} H^0_{\mathrm{dR}}(S^1) \xrightarrow{b} H^0_{\mathrm{dR}}(\mathbb{R}^0) \oplus H^0_{\mathrm{dR}}(\mathbb{R}^0) \xrightarrow{c} H^0_{\mathrm{dR}}(S^0) \xrightarrow{d} H^1_{\mathrm{dR}}(S^1) \xrightarrow{e} H^1_{\mathrm{dR}}(\mathbb{R}^0) \oplus H^1_{\mathrm{dR}}(\mathbb{R}^0)$$

$$0 \xrightarrow{a} H^0_{\mathrm{dR}}(S^1) \xrightarrow{b} \mathbb{R} \oplus \mathbb{R} \xrightarrow{c} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d} H^1_{\mathrm{dR}}(S^1) \xrightarrow{e} 0 \oplus 0$$

Figure 2. MV sequence for S^1.

Suppose by induction that up to some fixed n, $H^i_{\mathrm{dR}}(S^n) \cong \mathbb{R}$ for $i = 0$, $i = n$ and $H^i_{\mathrm{dR}}(S^n) \cong 0$ otherwise.

Now consider S^{n+1} with $A = S^{n+1} \setminus \{(0,0,\ldots,0,-1)\}$ and $B = S^{n+1} \setminus \{(0,0,\ldots,0,1)\}$. Note that again, $A \simeq B \simeq \mathbb{R}^0$, while $A \cap B \simeq S^n$. Considering the Mayer-Vietoris sequence in Figure 3, we have that x and z are both 0, so by exactness, y is an isomorphism, and hence $H^{i+1}(S^{i+1}) \cong \mathbb{R}$.

$$H^i(\mathbb{R}^0) \oplus H^i(\mathbb{R}^0) \xrightarrow{x} H^i(S^i) \xrightarrow{y} H^{i+1}(S^{i+1}) \xrightarrow{z} H^{i+1}(\mathbb{R}^0) \oplus H^{i+1}(\mathbb{R}^0)$$

$$0 \xrightarrow{\quad x \quad} \mathbb{R} \xrightarrow{y} H^{i+1}(S^{i+1}) \xrightarrow{\quad z \quad} 0$$

FIGURE 3. MV sequence for S^{n+1}.

If $i \neq n$, then the same sequence occurs, except that $H^i(S^j) \cong 0$ for $i \neq j$, and this forces $H^{i+1}(S^{j+1}) \cong 0$. ☐

Proposition 6.2. $H^i_{\mathrm{dR}}(T^n) \cong \mathbb{R}^{\binom{n}{i}}$.

Proof. First denote by $T^n = (S^1)^n$ the product of n circles. When $n = 0$, by convention we mean the space with one point $T^0 = \mathbb{R}^0$. Then considering that T^n may be defined inductively as $T^{n-1} \times S^1$ and applying Theorem 5.1, we have that

$$\begin{aligned} H^i(T^n) &= H^i(T^{n-1} \times S^1) \\ &\cong H^i(T^{n-1}) \oplus H^{i-1}(T^{n-1}) \end{aligned} \tag{1}$$

Using the calculation from Proposition 6.1, we have that $H^i(S^1) \cong \mathbb{R}$ for $i = 0, 1$ and $H^i(S^1) \cong 0$ otherwise. Compare the shape of Equation 1 to the shape of Pascal's Formula:

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$$

Hence the de Rham cohomology groups of T^n form a sort of pascals triangle, where each row has de Rham cohomology groups isomorphic to $\mathbb{R}^{\binom{n}{i}}$. This is partially demonstrated in Table 1

	H^0	H^1	H^2	H^3	H^4	H^5
T^0	$\mathbb{R}$	0	0	0	0	0
T^1	$\mathbb{R}$	$\mathbb{R}$	0	0	0	0
T^2	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}$	0	0	0
T^3	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^3$	$\mathbb{R}$	0	0
T^4	$\mathbb{R}$	$\mathbb{R}^4$	$\mathbb{R}^6$	$\mathbb{R}^4$	$\mathbb{R}$	0

TABLE 1. de Rham cohomology groups showing the Pascal's triangle pattern for the n-torus, $0 \leq n \leq 4$.

☐

References

[1] Bjørn Ian Dundas. *A Short Course in Differential Topology*. Cambridge University Press, 2018.
[2] John M Lee. *Smooth Manifolds*. Springer, 2012.
[3] Steven H Weintraub. *Differential Forms: Theory and Practice*. Elsevier, 2014.