MORSE FORMATIONS

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ABSTRACT. This paper collects various ideas I have in differential topology. Namely, it attempts to explore some connections between differential forms, Morse theory, and geometry. Specifically, generalizations of the gradient vector field of a function to a function defined on a manifold are defined, which is a critical concept for much of Morse theory, especially Morse homology. We also discuss using Morse theory to prove things about geometry, as well as what happens when a k-form is pulled back along the gradient of a Morse function. Of my original ideas herein, only a few probably hold up to any scrutiny, so be warned.

1. Introduction

1.1. **Notations.** In general, for f some sort of function between manifolds, f_* denotes the pushforward and f^* denotes the pullback. T_pM denotes the vector space of vectors tangent to some manifold M at a point p. v_p denotes a vector in T_pM. $\Omega^k(M)$ denotes the k dimensional smooth differential forms of M, or just $k-$forms for short.

1.2. **Notions.** The following are important definitions and theorems from basic Morse theory. In general, let $f : M \to \mathbb{R}$ be a smooth function.

Definition 1.1. *The **Hessian** is the (symmetric) matrix of second partials of f,*

$$H_f \triangleq \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

Definition 1.2. *A point $p \in M$ is called **degenerate** if $\det(H_f(p)) = 0$ and **non-degenerate** otherwise.*

Definition 1.3. *A function f with no degenerate critical points is called **Morse**.*

Definition 1.4. *The number of -1's in the diagonalized $H_f(p)$ is called the **index** of f at p a critical point of f. The index is usually denoted by λ.*

Theorem 1.5 (Morse Lemma)**.** *Let $f : M \to \mathbb{R}$ be a smooth function with non-degenerate critical point p_0. Then one may choose a local coordinate system $(X_1, X_2, \ldots, X_n)$ about p_0 such that f, when represented using these local coordinates, has the form*

$$f = -X_1^2 - \cdots - X_\lambda^2 + X_{\lambda+1}^2 + \cdots + X_n^2 + c$$

where λ is the index of f at p_0.

Corollary 1.6. *Non-degenerate critical points are isolated.*

Corollary 1.7. *A Morse function on a compact manifold admits only finitely many critical points.*

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2. Gradient-like Vector Fields

The following is a summary of section 2.3 from Matsumoto's Introduction to Morse Theory [Mat02].

Suppose M is a smooth manifold and $v_p \in T_p(M)$ is a tangent vector at a point $p \in M$.

Then one can write v_p using the 'differential standard basis' $\{\frac{\partial}{\partial x_i}\}$ as

$$v_p = \sum_i v_p^i \frac{\partial}{\partial x_i}$$

One reason for doing such a thing is to then define the directional derivative of a smooth function $f : M \to \mathbb{R}$ as

$$v_p \cdot f \triangleq \sum_i v_p^i \frac{\partial f}{\partial x_i}$$

To make things easier, we don't require our tangent vectors to be unit vectors, but if we did, then we get the interpretation that $v_p \cdot f$ is the amount that f changes in the direction of v_p.

Similarly, we can define a (smooth) vector field X as a function that sends a point $p \in M$ to a tangent vector at p, i.e. an element of $T_p(M)$.

$$X = \sum_i A_i \frac{\partial}{\partial x_i}$$

where the A_i are smooth functions.

A special kind of vector field for a function f defined on an open subset of $\mathbb{R}^n$ is

$$X_f \triangleq \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

i.e. a vector field where each coefficient is the partial derivative of f.

Observe that

$$X_f \cdot f = \sum_i \left(\frac{\partial f}{\partial x_i} \right)^2$$

hence $X_f \cdot f \geq 0$.

If an f is a Morse function, then locally about a critical point of index λ f can be expressed as

$$f = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2$$

and then

$$X_f = -2x_1 \frac{\partial}{\partial x_1} - \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_m \frac{\partial}{\partial x_m}$$

Observe that this can (so far) only be done locally in subsets of $\mathbb{R}^n$, but we would like a similar notion for manifolds. One way to achieve this is to take a function defined on a compact manifold, compute the gradient of a function on an open cover, and extend this smoothly to a vector field defined on the whole manifold. Such a vector field motivates the following definition:

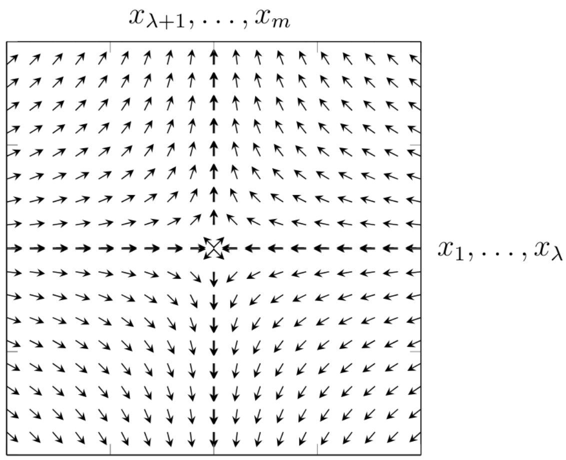

FIGURE 1. Gradient-like vector field at a critical point with index $1 < \lambda < m$

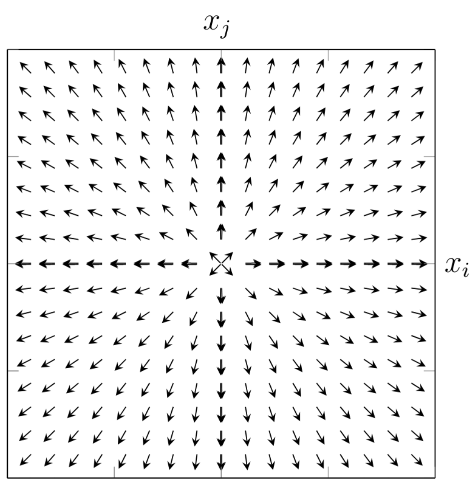

FIGURE 2. Gradient-like vector field at a critical point with index $\lambda = 0$

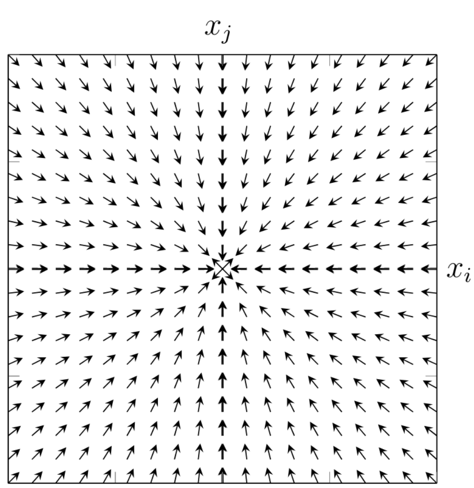

FIGURE 3. Gradient-like vector field at a critical point with index $\lambda = m$

Definition 2.1. *A **gradient-like vector field** X for a Morse function $f : M \to \mathbb{R}$ is a vector field such that*

1. $X \cdot f(p) > 0$ *for p not a critical point of f*
2. *If p is a critical point of index λ, then there exists a neighborhood V of p with a coordinate system $(x_1, \ldots, x_m)$ such that*

$$f = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 \cdots + x_m^2 + f(p)$$

and X can be written as its gradient vector field

$$X = -2x_1 \frac{\partial}{\partial x_1} - \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_m \frac{\partial}{\partial x_m} \quad (\flat)$$

Lemma 2.2 (Critical Point Cover)**.** *Suppose M for M a compact manifold, and f is a Morse function $f : M \to \mathbb{R}$. Then there exists a finite open covering $M = \cup_i U_i$ such that each critical point of f has a neighborhood contained in exactly one open U_i.*

Proof. Let $M = \cup_i U_i$ be a finite open covering. WLOG, suppose p_0 a critical point contained in $U_i \cap U_j$, two distinct open subsets in the open cover of M. Since the intersection of two open sets is open, one may pick a compact subset $V \subseteq U_i \cap U_j$ such that the interior $\mathrm{int}(V)$ contains p_0. Define $U_j' \triangleq U_j - V$. Then U_j' is open, because the compliment of U_j' is $U_j^c \cup V$ which is closed, which follows from the fact that compact implies closed in Hausdorff spaces, which M certainly is, being a smooth manifold. Now replace U_j by U_j' in the covering $\cup_i U_i$, which certainly still covers M since we only removed something from the intersection. $\square$

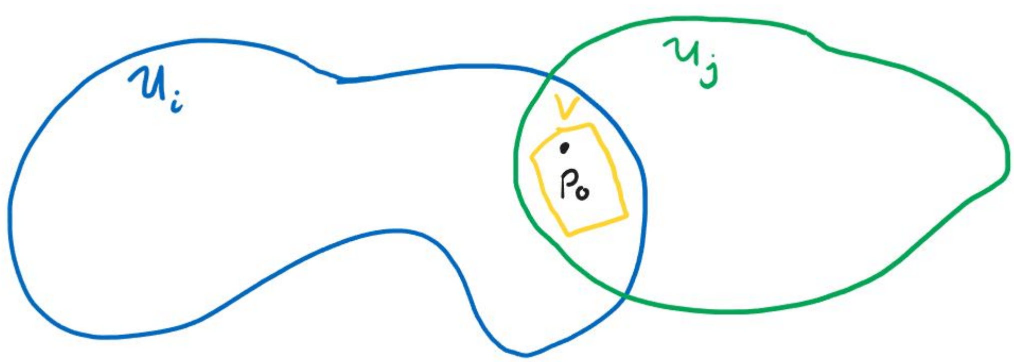

FIGURE 4. Two open subsets U_i and U_j in the open covering containing critical point p_0, together with containing compact V.

One may proceed inductively on the number of intersecting sets containing p_0. However, care must be taken in choosing the compact set V since removing V could potentially remove necessary points from another open neighborhood of another critical point, which may make that open neighborhood no longer open. The upshot is that because the critical points are isolated, it should be possible to remove small enough V and still be left with a neighborhood of each critical point.

One should also note that the converse is not true, i.e. there exist finite open covers with constituent open sets that do not contain any critical points of f.

Theorem 2.3 (Existence of Gradient-Like Vector Fields)**.** *If $f : M \to \mathbb{R}$ is a Morse function and M is a compact manifold, then there exists a gradient-like vector field for f on M.*

Proof. Begin by taking $M = \cup_i U_i$ the open cover of M and applying the Morse Lemma to each critical point to rewrite f locally in standard form. Further, apply Lemma 2.2 to each critical point to modify the open cover so that each critical point is contained a compact neighborhood which is contained in exactly one coordinate neighborhood U_i. If necessary, one may add in the necessary coordinate neighborhoods to get back to an open cover of M.

Now for each U_i, construct a gradient-like vector field using the coordinates of U_j

$$X_j \triangleq \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial}{\partial x_m}$$

Further, let $\{h_j\}$ be a smooth partition of unity with h_j corresponding to neighborhood U_j. Now define the vector field

$$X \triangleq \sum_{j=1}^{k} h_j X_j$$

where $h_j X_j$ has been extended to all of M by setting it to the zero vector outside of U_j.

Now, we wish to show that X is a gradient-like vector field for f on M. First, suppose p is not a critical point of f. Then $(h_j X_j \cdot f)(p) > 0$ if $p \in U_j$ and $(h_j X_j \cdot f)(p) = 0$ otherwise. Moreover, since the U_j cover M, every point is contained in some U_j, so X has at least one positive term for any p. Hence $X \cdot f > 0$ at non-critical points.

Second, suppose p_0 is a critical point of f. Recall f is in standard form in a compact neighborhood V about p_0, and $h_j X_j = X_j$ is the only nonzero term in X on V, so locally X has the form given by equation ($\flat$). □

To summarize, a smooth partition of unity subordinate to a compact covering of M was constructed to 'glue' together some local gradient vector fields into a global gradient-like vector field. Namely, if h_j is the smooth partition of unity and X_j is the gradient of f on compact K_j, then $X = \sum_j h_j X_j$, which is positive on non-critical points of M and has the form in ($\flat$) on critical points.

Definition 2.4 (Integral Curve)**.** *Suppose X is a vector field on manifold M. Then an integral curve is a function*

$$c_p : (-\infty, \infty) \to M$$

where $c_p(0) = p$ and

$$\frac{dc_p}{dt} = X_{c_p(t)}$$

In other words, this means that the velocity vector of the curve at a point in its image matches the vector field at that point. It is a fact that on compact manifolds without boundary, for arbitrary 'starting' point p, there exists an integral curve $c_p(t)$ for X. This fact can be concluded by applying the Picard–Lindelöf Theorem to the differential equation $\frac{dc_p}{dt}(t) = X_{c_p(t)}$ since $X_{c_p(t)}$ is a smooth function of t, and smooth functions have continuous derivative, hence are Lipschitz continuous.

The important thing that Matsumoto points out about integral curves of gradient-like vector fields is that they 'flow' between critical points, unless $c(0)$ is a critical point, in which case the integral curve is constant. This is useful because it helps show that roughly, the topology of a manifold remains constant between critical points.

Define the submanifold (with boundary) from a to b for Morse $f : M \to \mathbb{R}$

$$M_{[a,b]} \triangleq \{p \in M \mid a \leq f(p) \leq b\}$$

If f is the height function of M, then $M_{[a,b]}$ is just the points of M with height between a and b.

Theorem 2.5. *If f is a Morse function that has no critical value in the interval $[a, b]$, then $M_{[a,b]}$ is diffeomorphic to $f^{-1}(a) \times [0, 1]$.*

Proof. Indeed, since f has no critical values in $[a,b]$ then if X is a gradient-like vector field for f,

$$Y \triangleq \frac{1}{X \cdot f}$$

is a vector field defined on $M_{[a,b]}$.

Let $c_p(t) = (x_1(t), \ldots, x_N(t))$ be an integral curve of Y starting at $p \in f^{-1}(a)$. Then

$$\begin{aligned}
\frac{d}{dt} f(c_p(t)) &= \sum_{j=1}^{N} \frac{dx_j}{dt}(t) \frac{\partial f}{\partial x_j}(c_p(t)) \\
&= \left(\frac{dx_1}{dt}(t), \ldots, \frac{dx_N}{dt}(t) \right) \cdot f \\
&= Y_{c(t)} \cdot f \\
&= \frac{1}{X \cdot f}(X \cdot f) \\
&= 1
\end{aligned}$$

so that $c_p(t)$ moves at a unit pace w.r.t. f, meaning that $c_p(b-a) \in f^{-1}(b)$ since the length of $[a,b]$ is $b-a$.

Now let $h : f^{-1}(a) \times [0, b-a] \to M_{[a,b]}$ be defined as

$$h(p,t) \triangleq c_p(t)$$

which is a diffeomorphism since $c_p(t)$ is smooth with respect to p and with respect to t, and since integral curves are unique here. Hence $M_{[a,b]} \cong f^{-1}(a) \times [a,b]$. □

In fact, this can be extended to a statement about smooth functions (not necessarily Morse) for which 0 is not a critical value, since application of the implicit function theorem[1] says that $f^{-1}(0)$ is a submanifold of m-manifold M with dimension $m-1$.

The main thrust of this section is that later on in the text, and in Morse theory in general, the topology of a manifold is studied by breaking up the manifold between critical points using these 'collar neighborhoods' which look like a slice of the manifold cross an interval. Equivalently, these are sections of the manifold where the topology cannot change, because they contain no critical points for the corresponding Morse function f. This leads to things like Morse homology and the Morse inequalities.

[1]This is using the version of the implicit function theorem described in Matsumoto where one of the m coordinates can be solved for the others using the relation $f(x_1, \ldots, x_m) = 0$

3. How tall is your manifold?

In this section we describe why height functions of manifolds are the only Morse functions worth considering.

Proposition 3.1. *All Morse functions can be expressed as the height function of some manifold.*

Proof. If $f : M \to \mathbb{R}$ is a Morse function of M, then $F(\vec{x}, f(\vec{x}))$ is a smooth function from M to $M \times \mathbb{R}$. Indeed, $M \times \mathbb{R}$ is a smooth manifold, so then composing projection with F is a Morse function, since $\pi_{n+1} \circ F = f$. Further, π_{n+1} is the height function for the manifold $M \times \mathbb{R}$. □

4. Do critical points curve?

In this section, we investigate computing Gaussian curvature of Morse functions $f : \mathbb{R}^2 \to \mathbb{R}$ specifically at critical points. Moreover, we investigate applying Morse theory and these connections between critical points and curvature to compact geometries in $\mathbb{R}^3$.

4.1. **Curvature.** The Gaussian curvature of surface S with local coordinates (x, y) is $A_{2,0}A_{0,2} - A_{1,1}^2$ [HH98]. The Taylor Series for this function is given by $f(x, y) = 1/2(A_{2,0}x^2 + 2A_{1,1}xy + A_{0,2}^2 y^2) +$ higher terms.

$$A_{2,0} = \frac{\partial^2 f}{\partial x^2}$$

$$A_{1,1} = \frac{\partial^2 f}{\partial x \partial y}$$

$$A_{0,2} = \frac{\partial^2 f}{\partial y^2}$$

The mean curvature is $1/2(A_{2,0} + A_{0,2})$.

Observe that the Gaussian curvature is precisely the determinant of the Hessian matrix for f at p.

Proposition 4.1. *Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then if $p \in \mathbb{R}^2$ is a non-degenerate critical point of f, then the Gaussian curvature at $(p, f(p))$ on the surface implicitly defined by $F(x, y, z) = z - f(x, y) = 0$ is nonzero, $K(F) \mid_{(p,f(p))} \neq 0$.*

Proof. Indeed, for $F(x, y, z) = z - f(x, y) = 0$, $F_z = \frac{\partial F}{\partial z} = 1 \neq 0$. Observe that $\frac{\partial f}{\partial x} = -\frac{F_x}{F_z} = -F_x$ and $\frac{\partial f}{\partial y} = -\frac{F_y}{F_z} = -F_y$. Suppose that $p \in S$ is a non-degenerate critical point of $f(x, y)$. Then $F_x = F_y = 0$. Further, $H(f)$ is nonsingular, namely $|H(f)| \neq 0$.

According to [Gol05] equation 4.1, the following formula for the Gaussian curvature of an implicitly defined surface $F(x, y, z) = 0$ is true:

$$K(F) = -\frac{\begin{vmatrix} H(F) & \nabla F^T \\ \nabla F & 0 \end{vmatrix}}{(\nabla F)^4}$$

where $H(F)$ is the Hessian matrix of F and ∇F is the gradient of F.

Now to compute the Hessian matrix f,

$$H(f) = \frac{1}{F_z^2} \begin{bmatrix} F_{xx}F_z - F_xF_{zx} & F_{xy}F_z - F_xF_{zy} \\ F_{xy}F_z - F_xF_{zy} & F_{yy}F_z - F_yF_{zy} \end{bmatrix}$$

However, evaluated at the critical point p, we can use the fact that $F_x = F_y = 0$ to reduce $H(f)$ to

$$H(f) = \frac{1}{F_z^2} \begin{bmatrix} F_{xx}F_z & F_{xy}F_z \\ F_{xy}F_z & F_{yy}F_z \end{bmatrix}$$

Since p was assumed non-degenerate, $|H(f)| \neq 0$ so $F_{xx}F_{yy} - F_{xy}^2 \neq 0$. Observe $(\nabla F)^4 K(F) = F_{xx}F_{yy} - F_{xy}^2$, and $\nabla F = [0,0,1]$, so $(\nabla F)^4 = 1$ and $K(F) \neq 0$. $\square$

Proposition 4.2. *All closed smooth manifolds have a Morse function with at least one critical point.*

Proof. Let M be a closed smooth m-manifold. Indeed, the constant function c_0 is a smooth function on M. Locally in a coordinate neighborhood of M, linearly morsify the constant function, yielding $a_1x_1 + \cdots + a_nx_n + c_0$, which has no critical points. Now consider the standard bump function $f(x)$, but where only $f(0) = 1$, and everywhere else $f(x) < 1$. Since $f(x)$ is smooth, it has bounded derivative, so pick a_i above smaller than the max derivative.

Then $F(x) \triangleq f(x) + a_1x_1 \cdots + a_nx_n + c_0$ is a Morse function with exactly two critical points, one in the 'elbow' of the bump function and one on the upward facing portion of the 'hump' of the bump function. Computing derivatives of $F(x)$, one sees the critical points are equivalently points $p \in M$ where $\frac{\partial f}{\partial x_i} = -a_i$ for all i. Further, neither of these critical points are degenerate, because the Hessian matrix for $F(x)$ is the same as $f(x)$, and the Hessian of $f(x)$ is nonsingular on the support of $f(x)$. $\square$

Corollary 4.3. *All closed smooth surfaces embedded in $\mathbb{R}^3$ have a point with nonzero Gaussian curvature. Namely, there are no closed surfaces with constant 0 curvature.*

This follows from combining the previous two propositions; namely, we take a compact surface embedded in $\mathbb{R}^3$, define a Morse function where the compact surface is (locally) defined by the graph of the Morse function[2], and then compute Gaussian curvature as above to see that the surface had to be closed in order to be compact.

The reason we need to be embedded in $\mathbb{R}^3$ is because the flat torus is a counterexample to this fact in higher dimensions. The flat torus is a closed smooth surface with zero curvature at every point, but it must be embedded in $\mathbb{R}^4$.

5. Why not Morse Forms?

In this section we discuss half-hearted (read: failed) attempts at defining what a Morse form could be. The main thrust of this section, however, is *why* these attempts fail to be fruitful.

Consider the following potential definition for a *Morse (differential) form*:

Definition: A Morse form in $\mathbb{R}^n$ is a differential form $\omega = A_1 dx_{I_1} + \cdots + A_m dx_{I_m}$ where all A_i are Morse functions.

The problem with this is that points where A_i has (nondegenerate) critical points might differ from where A_j has (nondegenerate) critical points. However, if one is willing to swallow this complication, then finding nondegenerate critical points of ω becomes solving a (homogeneous) system of Hessian matrix determinants $\{\det H(A_i)(p)\}$.

An alternative algorithm would be to search the space of critical points associated to one H_{A_i} to see if $\det H_{A_j}(p) \neq 0$ for any j. Once a certain point has been eliminated as a possibility for H_{A_i}, it can be ignored for the rest of the matrices and hence one only has to check all critical points for one Hessian.

Definition 5.1. *The Jacobian J is the n by m matrix of partial derivatives*

$$J(k)_{ij} = \left[\frac{\partial k_i}{\partial u_j}\right]$$

The pushforward of $k : M \to N$ for M an m dimensional manifold and N an n dimensional manifold, $k_* : T_pM \to T_pN$, is a linear operator generalizing the Jacobian between subsets of R^m and R^n. This pushforward k_* maps tangent vectors to tangent vectors.

[2]This should actually be proven.

Note that pushing forward tangent vectors only works pointwise, i.e. it is not possible to push forward a vector field, because the map k might not be injective, meaning multiple tangent vectors in the tangent bundle of M, TM, might get mapped to the same $T_k(p)N$, which would mean a vector field would not be sent to a vector field.

5.1. **The Hessian.** Note that for $f : \mathbb{R}^n \to \mathbb{R}$, then the Hessian matrix H_f can be expressed as the Jacobian of the gradient of f, $J(\nabla f)$. So $H_f = J(\nabla f)$.

This Jacobian expresses the pushforward of tangent vectors along the gradient of f.

$$(\nabla f)_* : v_p \mapsto (J(\nabla f))(v_p)$$

Now (loosely) considering ∇f as a map from $\mathbb{R}^n \to \mathbb{R}^n$, one can consider $\omega \in \Omega^n(\mathbb{R}^n)$ a nonzero n-form on $\mathbb{R}^n$ with compact support, and think about the following integral:

$$\begin{aligned}\int_{\mathbb{R}^n} (\nabla f)^*(\omega) &= \int_{\mathbb{R}^n} \omega(\nabla f)_* \\ &= \int_{\mathbb{R}^n} \omega J(\nabla f) \\ &= \int_{\mathbb{R}^n} \sum_I a_I \det(H_f) dx^{i_1} \dots dx^{i_k}\end{aligned}$$

The interesting thing about this integral, however, is that the $\det(H_f)$ shows up, which is a crucial condition for degeneracy of critical points. Moreover, recall that the Morse Lemma implies that critical points of a Morse function are isolated, and since isolated sets are countable, the set of critical points of a Morse function is measure 0. Hence, the value of the integral only comes from the non-critical points of $\mathbb{R}^n$. Further, as a consequence, if $\det(H_f(p)) = 0$ for all non-critical points of a manifold for Morse function f, then $\int_{\mathbb{R}^n} (\nabla f)^*(\omega) = 0$.

We will now show that this is a contradiction when ∇f is a diffeomorphism and hence there are no such Morse functions with $\det(H_f(p)) = 0$ for all non-critical points.

Without loss of generality, pick ω that integrates to a nonzero value over $\mathbb{R}^n$. Then, by the definition of integrating a differential form, and since we're assuming $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism,

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} (\nabla f)^* \omega$$

But then $\int_{\mathbb{R}^n} (\nabla f)^* \omega \neq 0$ so there is a contradiction above. Hence there do not exist Morse functions with diffeomorphism gradients where all non-critical points are degenerate. In other words, for every such Morse function, there exists at least one non-critical point p where $\det(H_f(p)) \neq 0$.

There is an alternative way to see a stronger result, due to Michael Moy. The stronger result is that $\det(H_f(p)) = 0$ can't be true for all points except the critical points (if there are any) Since f is smooth, H_f is smooth as a matrix of functions, and det of a matrix of smooth functions is a polynomial of smooth functions, and hence $\det(H_f)$ is smooth. Further, since $\det(H_f(p)) \neq 0$ for critical points, we have that $\det(H_f(p)) \neq 0$ in a neighborhood around each critical point, since it needs to be smooth, and hence continuous.

Yet another way to imagine this situation being true is by thinking again about the Morse Lemma, which would imply that locally a Morse function about a critical point looks like either a hyperboloid or parabolic, and points in that neighborhood have nonzero $\det(H_f(p))$.

Regardless, this implies the existence, for an arbitrary smooth manifold M, of Morse functions f that have critical points which have surrounding regions of non-degenerate points (sort of like continents), themselves separated by regions of degenerate points - the union of which I will choose to call the **degeneracy locus** (sort of like oceans). With this idea in mind, we get the following illustration of such a Morse function on the sphere. One can think of such a function as a sum of bump functions from the sphere such that the bump function is morse, i.e. only one point has value $h(p) = 1$, otherwise the critical points would not be isolated. This continent and ocean idea is illustrated in the following figure.

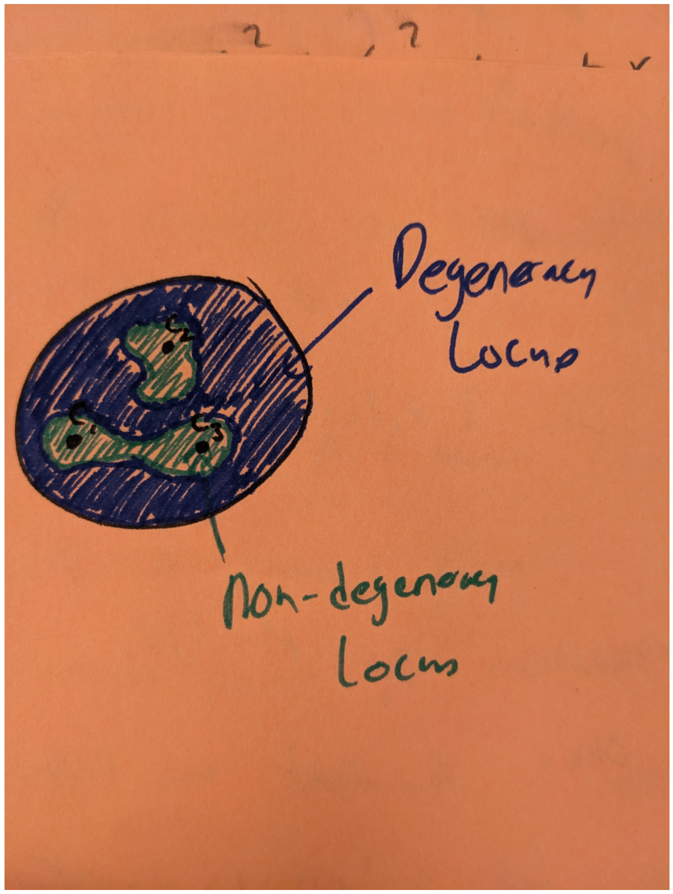

Figure 5. Degeneracy Locus Sketch

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