KNESER'S PROOF OF THE FOUR VERTEX THEOREM

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Consider a plane curve C with smoothly varying tangent and curvature. If $a, b \in C$ with the radius of curvature monotonically decreasing from a to b, let E be the associated evolute to C and let $a_1, b_1 \in E$ be the centers of the osculating circles at a and b, respectively. Then the arc-length from a_1 to b_1 is equal to the difference of the curvatures at a and b: arc $a_1b_1 = aa_1 - bb_1$. Since the evolute cannot contain any straight segments, the straight-line distance

$$a_1b_1 < aa_1 - bb_1. \tag{1}$$

Suppose the osculating circles at a and b intersect at a point c; then, by considering the triangle a_1b_1c, it's easy to see that $a_1b_1 > a_1c - b_1c = aa_1 - bb_1$, contradicting (1). On the other hand, if neither of the osculating circles encloses the other, then $a_1b_1 > aa_1 + bb_1$, which also contradicts (1). Therefore, the larger of the osculating circles (i.e. the one at a) completely encloses the smaller. Since the only assumption necessary for this conclusion was that the radius of curvature was monotonically decreasing, we see that the region between the osculating circles at a and b is filled by non-intersecting osculating circles corresponding to points along the arc from a to b.

Now, suppose C violates the Four Vertex Theorem; that is, suppose the radius of curvature has a minimum at b and a maximum at a. Then the result proved above holds along both arcs from a to b, so each point on C (except a and b) lies on exactly two osculating circles. In what follows, we will show that this is impossible, and so such a C cannot exist.

To that end, stereographically project C to a sphere tangent to the plane of C; let C_0 denote the image of C. Since stereographic projection is conformal, it's easy to see that osculating circles to C are mapped to osculating circles to C_0. Moreover, since C intersects any circle in an even number of points (counting multiplicity) and since any circle on the sphere defines a plane, we see that C_0 intersects any plane in an even number of points (counting multiplicity).

Now, if $p \in C_0$, P_1 is a plane through p not tangent to C_0 and P_2 is a plane parallel to T_pS^2, let C_{01} be the image of C_0 under stereographic projection to P_2 (i.e. p acts like the north pole in the usual stereographic projection). Since P_1 intersects C_0 in an odd number of points aside from p, the image of those points in C_{01} lie on a straight line; by varying P_1, we see that C_{01} intersects any line in an odd number of points. Thus, C_{01} is an "odd circuit".

A theorem of Möbius says that any odd circuit has at least 3 inflection points, so we see that C_{01} has at least 3 inflection points. However, an inflection point on C_{01} corresponds to a point on C_0 whose osculating circle passes through p, so we see that p lies on at least 4 osculating circles to C_0 (counting its own). Since our choice of p was arbitrary, this in turn implies that every point on C lies on at least four osculating circles, which gives the necessary contradiction to conclude that such a C cannot exist.

Note that, as stated, Kneser's proof does not cope well with mixed positive and negative curvature. However, this ceases to be an issue if we stereographically project to the sphere *before* making the argument that each point lies on exactly two osculating circles, so this does indeed prove the Four Vertex Theorem.

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