REMARKS ON THE NUMBER OF EXTREMES OF THE CURVATURE ON CLOSED CURVES AND ON RELATED QUESTIONS IN NON-EUCLIDEAN GEOMETRY

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[Translator's note: This is a translation by Clay Shonkwiler of the first section of Adolf Kneser's paper "Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nicht-euklidischen Geometrie", which was originally published in the 1912 Festschrift of Heinrich Weber. Since I don't actually speak German, I've taken some liberties in translating (e.g. in the second-to-last paragraph I've assigned names to the planes under consideration), so this might more accurately be termed an interpretation of Kneser's paper which maintains its mathematical content, rather than a strict translation.]

Mr. Carathéodory recently drew my attention to the fact that the curvature of a simple closed curve always has at least two maxima and two minima. Thus was I reminded of some geometrical investigations which I published in the 31st, 34th and 41st volumes of *Mathematischen Annalen*, and it turned out that this fact was easy to prove in a completely descriptive way. The geometrical views which lead to the proof can be extended to the case where one has hyperbolas with fixed asymptotic directions in place of the circle and uses the form $\sqrt{dx^2 - dy^2}$ for the length element in rectangular coordinates x, y. Such a non-Euclidean geometry has recently been considered by physicists following the Lorentz-Einstein relativity principle; the resulting geometry may therefore garner a certain interest.

I.

Let $\mathfrak{C}$ be a simple closed plane curve of smoothly varying tangent and curvature, meaning we can apply the standard theory of the evolute. If AB is a segment of the curve on which the radius of curvature monotonically decreases from A to B, let A_1B_1 be the corresponding segment of the evolute. The length of the latter is equal to the difference of the radii of curvature at the points A and B:

$$\mathrm{arc} A_1B_1 = AA_1 - BB_1,$$

and, since the evolute can have no straight sections, it follows that

$$A_1B_1 < AA_1 - BB_1. \tag{1}$$

Now suppose the osculating circles at A and B (with centers at A_1 and B_1 and radii AA_1 and BB_1, respectively) intersect at a point C. Then we would have the following inequality from the triangle A_1B_1C:

$$A_1B_1 > A_1C - B_1C$$

or

$$A_1B_1 > AA_1 - BB_1,$$

which contradicts (1). Therefore, the osculating circles at A and B cannot intersect. Moreover, one of these circles must contain the other, else it would follow that

$$A_1B_1 > AA_1 + BB_1,$$

which also contradicts (1). Thus, the smaller of the two circles, namely the one centered at B_1, must be completely contained in the larger. It follows easily that the osculating circles along the arc AB completely fill (foliate?) the region between the circles associated to the endpoints.

Now, maintaining the smoothness conditions on the curve from above, let B be the point of minimum radius of curvature and suppose the radius of curvature monotonically increases to a point C. Then, as above, the osculating circles along the arc BC fill the region between the circles at the endpoints. If the points A and C coincide and describe the point of maximum radius of curvature, then the curve $ABCA$ has exactly one largest and one smallest circle of curvature with the region between these two circles covered exactly twice, by the circles along the arcs AB and BC. The curve ABC runs along the boundary of this region and each point on the curve other than A and B lies on exactly two osculating circles: the one corresponding to that point and another from the opposite arc.

This, however, is impossible given our conditions on $\mathfrak{C}$.

In order to see this, stereographically project $\mathfrak{C}$ to a sphere sitting on the plane of the curve. Then each circle in the plane projects to a spherical circle; if $\mathfrak{C}_0$ is the image of $\mathfrak{C}$, then each osculating circle of $\mathfrak{C}$ maps to a circle on the sphere which has 3-point contact with $\mathfrak{C}_0$ and so lies in one of the osculating planes of $\mathfrak{C}_0$. The result proved for $\mathfrak{C}$ requires that, except at A and B, the osculating plane to each point of $\mathfrak{C}_0$ coincides with the osculating plane of another point on the curve. The osculating planes to A and B obviously have 4-point contact with $\mathfrak{C}_0$.

Now the curve $\mathfrak{C}$ will, because it is closed, intersect any circle in its plane in an even number of points; hence, $\mathfrak{C}_0$ intersects any plane in an even number of points. If P is a point on $\mathfrak{C}_0$, let $\mathfrak{P}_1$ be a plane though P and $\mathfrak{P}_2$ a plane not passing through P (NB: it's important in what follows that $\mathfrak{P}_1$ is not tangent to the curve and that $\mathfrak{P}_2$ is actually parallel to the tangent plane to the sphere at P, neither of which Kneser assumes). Then $\mathfrak{P}_1$ intersects $\mathfrak{C}_0$ in an odd number of points aside from P. If we stereographically project $\mathfrak{C}_0$ from P to $\mathfrak{P}_2$ (yielding a curve $\mathfrak{C}_{01}$), then this implies that $\mathfrak{C}_{01}$ has an odd number of points lying on the line $\mathfrak{P}_1 \cap \mathfrak{P}_2$. Since our choice of $\mathfrak{P}_1$ was arbitrary, we see that $\mathfrak{C}_{01}$ intersects any line in an odd number of points, so

$\mathfrak{C}_{01}$ is an "odd circuit" in the sense of Staudts. This curve is free of double points (i.e. simple), since such could exist only if a ray from P intersected $\mathfrak{C}_0$ in two points aside from P, which is impossible since $\mathfrak{C}_0$ lies on a sphere. Furthermore, the curve $\mathfrak{C}_{01}$ has, so long as it is finite, continuous tangents and curvature and, where it is infinite, they are continuous in the projective sense.

To the curve $\mathfrak{C}_{01}$ we now apply a theorem proved by Möbius in an investigation of third-degree curves (*Works* vol. II) which says that an odd circuit free of double points must always have at least three points of inflection. This immediately leads to a contradiction with our assumption that $\mathfrak{C}$ has only two extreme values of the radius of curvature. Indeed, Möbius' theorem implies that at least three osculating planes associated to points other than P must contain P, so at least three osculating circles other than its own must pass through a point of $\mathfrak{C}$. This contradiction demonstrates that our assumption about $\mathfrak{C}$ is untenable: on any simple closed curve the radius of curvature must have at least two maxima and two minima.