VARIATIONAL CHARACTERIZATION OF A SUM OF CONSECUTIVE EIGENVALUES; GENERALIZATION OF INEQUALITIES OF PÓLYA-SCHIFFER AND WEYL

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Abstract. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ be the eigenvalues of a vibrating system, an extremal property of $\sum_1^n \lambda_i$ and $\sum_1^n \lambda_i^{-1}$, suggested by the work of Pólya-Schiffer [1], is established and generalized to $\sum_{k+1}^{k+n} \lambda_i$ and $\sum_{k+1}^{k+n} \lambda_i^{-1}$: on the one hand in the sense of Poincaré, on the other in the sense of the "Max-Min" property of Courant-Weyl. We establish inequalities which reduce to those of Pólya-Schiffer [1] for $k = 0$ and to those of Weyl [2] for $n = 1$.

1. Definition of the "Rayleigh Trace" $TR[L_n]$ on a linear space L_n and of the "Trace Inverse" $TRinv[L_n]$

We consider two positive definite quadratic forms $A(\nu, \nu)$ and $B(\nu, \nu)$ [3] on a vector or functional space; the Rayleigh quotient will be $R[\nu] = \frac{A(\nu,\nu)}{B(\nu,\nu)}$. We will suppose that the beginning of the spectrum is discrete.

Given a linear subspace L_n of dimension n, choose n vectors $\nu_1, \dots, \nu_n$ which are pairwise orthogonal in the metric $B : B(\nu_i, \nu_j) = 0$ if $i \neq j$; we define

$$TR[L_n] = R[\nu_1] + \cdots + R[\nu_n]. \tag{1}$$

This is the trace of the matrix associated with A in L_n under the metric B: thus this definition is independent of the choice of $\nu_1, \dots, \nu_n$.

Now, choose n vectors $\omega_1, \dots, \omega_n \in L_n$ pairwise orthogonal in the metric $A : A(\omega_i, \omega_j) = 0$ if $i \neq j$; we define

$$TRinv[L_n] = \frac{1}{R[\omega_1]} + \cdots + \frac{1}{R[\omega_n]}. \tag{2}$$

This is the trace of the matrix associated with B in L_n under the metric A: thus this definition is independent of the choice of $\omega_1, \dots, \omega_n$.

2. Variational characterization of $\sum_1^n \lambda_i$ and of $\sum_1^n \lambda_i^{-1}$

We part from the recurrent definition of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ and of the corresponding eigenvectors $u_1, u_2, u_3, \dots$:

$$\lambda_1 = \min_{\nu} R[\nu] = R[u_1]; \quad \lambda_2 = \min_{B(u_1,\nu)=0} R[\nu] = R[u_2]; \quad \cdots$$

For any $n \geq 1$,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \min_{\text{choice of } L_n} TR[L_n]. \tag{3}$$

In effect, there exists in all L_n:

a vector ν_n which is B-orthogonal to $u_1, \ldots, u_{n-1}$, so $R[\nu_n] \geq \lambda_n$;
a vector ν_{n-1} which is B-orthogonal to $u_1, \ldots, u_{n-2}$ and to ν_n, so $R[\nu_{n-1}] \geq \lambda_{n-1}$;
$\vdots$
a vector ν_1 which is B-orthogonal to $\nu_n, \ldots, \nu_2$, and $R[\nu_1] \geq \lambda_1$.

In the sum: $\lambda_1 + \cdots + \lambda_n \leq TR[L_n]$. In addition,

$$TR[L(u_1, \ldots, u_n)] = \lambda_1 + \cdots + \lambda_n;$$

(3) follows. Similarly,

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} = \max_{\text{choice of } L_n} TRinv[L_n]. \tag{4}$$

3. Recurrent characterization of $\sum_{k+1}^{k+n} \lambda_i$ and $\sum_{k+1}^{k+n} \lambda_i^{-1}$:

$$\sum_{k+1}^{k+n} \lambda_i = \min_{\text{choice of } L_n \ B\text{-orthogonal to } L(u_1,\ldots,u_k)} TR[L_n]. \tag{5}$$

In effect: in all *tel* L_n there exists a vector ν_{k+n} B-orthogonal to $u_1, \ldots, u_{k+n-1}$, therefore $R[\nu_{k+n}] \geq \lambda_{k+n}$; etc.

$$\sum_{k+1}^{k+n} \frac{1}{\lambda_i} = \max_{\text{choice of } L_n \ A\text{-orthogonal to } L(u_1,\ldots,u_k)} TRinv[L_n]. \tag{6}$$

4. Direct characterizations of $\sum_{k+1}^{k+n} \lambda_i$ and $\sum_{k+1}^{k+n} \lambda_i^{-1}$

4.1. Extremal property "in the style of Poincaré".

$$\sum_{k+1}^{k+n} \lambda_i = \min_{\text{choice of } L_{k+n}} \max_{\text{choice of } L_n \subset L_{k+n}} TR[L_n]; \tag{7}$$

$$\sum_{k+1}^{k+n} \frac{1}{\lambda_i} = \max_{\text{choice of } L_{k+n}} \min_{\text{choice of } L_n \subset L_{k+n}} TRinv[L_n]. \tag{8}$$

4.2. **Extremal property "in the style of Courant-Weyl".**

$$\sum_{k+1}^{k+n} \lambda_i = \max_{\text{choice of } L_k} \min_{\text{choice of } L_n \ B\text{-orthogonal to } L_k} TR[L_n]; \tag{9}$$

$$\sum_{k+1}^{k+n} \frac{1}{\lambda_i} = \min_{\text{choice of } L_k} \max_{\text{choice of } L_n \ A\text{-orthogonal to } L_k} TRinv[L_n]. \tag{10}$$

5. Generalized inequalities of Pólya-Schiffer [1] and of Weyl [2]

5.1. **Schrödinger-type equation.** $\nabla u + [\lambda - W(x,y,z)]u = 0$ with certain fixed conditions on limits;

$$R^{(W)}[\nu] = \frac{D(\nu) + \iiint W\nu^2 d\tau}{\iiint \nu^2 d\tau},$$

where $d\tau$ is the volume element and $D(\nu)$ is the Dirichlet integral.

$$\sum_{i=1}^{n} (\lambda_{k_1+i}^{(W_1)} + \lambda_{k_2+i}^{(W_2)} - 2\lambda_{k_1+k_2+i}^{[(W_1+W_2)/2]}) \le 0 \quad (k_1 \ge 0, k_2 \ge 0, n \ge 1). \tag{11}$$

Proof. Denote by $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \ldots$ the eigenfunctions of $\widetilde{W}(x,y,z) = (W_1 + W_2)/2$; in $L(\widetilde{u}_1, \ldots, \widetilde{u}_{k_1+k_2+n})$, there is an L_n orthogonal to both $L(u_1^{(W_1)}, \ldots, u_{k_1}^{(W_1)})$ and $L(u_1^{(W_2)}, \ldots, u_{k_2}^{(W_2)})$; thus, under the conditions of paragraph 3,

$$\sum_{k_1+1}^{k_1+n} \lambda_i^{(W_1)} + \sum_{k_2+1}^{k_2+n} \lambda_i^{(W_2)} \le TR^{(W_1)}[L_n] + TR^{(W_2)}[L_n] = 2TR^{(\widetilde{W})}[L_n] \le 2 \sum_{k_1+k_2+1}^{k_1+k_2+n} \lambda_i^{(\widetilde{W})}.$$

□

For $k_1 = k_2 = 0$, we have a convex inequality of the type of Pólya-Schiffer [1]; for $n = 1$, we have an inequality of the type of Weyl [2].

5.2. **Inhomogeneous vibrating system.** $\mathfrak{L}[u] - \lambda\rho(x,y,\ldots)u = 0$ with certain fixed conditions on the boundary, and with density $\rho \ge 0$. (Here $\mathfrak{L}$ is a self-adjoint linear differential operator). The Rayleigh quotient is $R^{(\rho)}[\nu] = \int \nu\mathfrak{L}[\nu]d\tau / \int \rho\nu^2 d\tau$.

$$\sum_{i=1}^{n} \left(\frac{1}{\lambda_{k_1+1}^{(\rho_1)}} + \cdots + \frac{1}{\lambda_{k_N+i}^{(\rho_N)}} - \frac{1}{\lambda_{k_1+\cdots+k_N+i}^{(\rho_1+\cdots+\rho_N)}} \right) \ge 0. \tag{12}$$

For $N = 2$ and $k_1 = k_2 = 0$, this gives a convex inequality of the type of Pólya-Schiffer [1]; for $n = 1$, this gives an inequality of the type of Weyl [2]. If $k_1 = k_2 = \ldots = k_N = 0$ and $n = \infty$, there is equality; we will return to this.

References

[1] G. Pólya and M. Schiffer, *J. Anal. Math.*, 3 (2nd part), 1953-1954, p. 245-345, in particular p. 286-290.

[2] H. Weyl, *Math. Ann.*, 71, 1912, p. 441-479, in particular p. 445; also see J. Hersch, *Propriétés de convexité du type de Weyl pour des problèmes de vibration et d'équilibre*, to appear in *Z.A.M.P.*

[3] We always suppose B is positive definite; the relations (1), (3), (5), (7), (9), (11) remain valid if A is indefinite with only finitely many negative eigenvalues.

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