FOUR ISOPERIMETRIC PROPERTIES OF HOMOGENEOUS SPHERICAL MEMBRANES

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Abstract. With the help of a "conformal mapping" [1] of the coordinate functions $\hat{x}, \hat{y}, \hat{z}$ of the sphere, we will show four inequalities for inhomogeneous vibrating membranes: the first on a surface of the type of a sphere, the second on a Jordan domain, the third on a "bilatere" and the fourth on a "trilatere" [2].

1

Let S be a closed surface of the topological type of a sphere which is conformal, except in isolated points, to the sphere $\hat{S} : \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 1$, $P \in S$ if and only if $\hat{P}(\hat{x}, \hat{y}, \hat{z}) \in \hat{S}$; on S define the function $\rho(P) \geq 0$ (the density); $M = \oint\oint_S \rho dS$ is the "total mass". If Δ is the Laplacian in the sense of Beltrami, the differential equation $\Delta u(P) + \mu\rho(P)u(P) = 0$ on S has the eigenvalues $\mu_1 = 0 < \mu_2 \leq \mu_3 \leq \mu_4 \leq \ldots$. By excluding the trivial case of point masses, we obtain

$$\left(\frac{1}{\mu_2} + \frac{1}{\mu_3} + \frac{1}{\mu_4}\right)\frac{1}{M} \geq \frac{3}{8\pi} \approx 0.11937. \tag{1}$$

There is equality when $\rho =$ constant on any sphere of radius R (then $\mu_2 = \mu_3 = \mu_4 = 2/R^2\rho$ and $M = 4\pi R^2\rho$); and on any S with sufficient density $\rho(P)$.

Proof. Suppose $\hat{\rho}(\hat{P}) = \rho(P)(dS/d\hat{S})$. If the centre of mass $\hat{G}$ of the masses $\hat{\rho}$ coincides with the center $\hat{O}$ of the sphere $\hat{S}$, then the images $X(P)$, $Y(P)$, $Z(P)$ of $\hat{x}, \hat{y}, \hat{z}$ on S satisfy $\oint\oint_S \rho X dS = \oint\oint_{\hat{S}} \hat{\rho}\hat{x} d\hat{S} = 0$ and, similarly, $\oint\oint_S \rho Y dS = 0$, $\oint\oint_S \rho Z dS = 0$. Moreover, by the conformal invariance of the mixed Dirichlet integral, $D_S(X, Y) = D_{\hat{S}}(\hat{x}, \hat{y}) = 0$ (similarly $D_S(Y, Z) = D_S(Z, X) = 0$) and $D_S(X) = D_{\hat{S}}(\hat{x}) = 8\pi/3$ (and similarly $D_S(Y) = D_S(Z) = 8\pi/3$). Also, $X^2 + Y^2 + Z^2 \equiv \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \equiv 1$. The linear space $L(X, Y, Z)$ generated by the functions $X(P)$, $Y(P)$, $Z(P)$ is admissible for the variational characterization [3] of $\mu_2^{-1} + \mu_3^{-1} + \mu_4^{-1}$:

$$\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_4} \geq \frac{\oint\oint_S \rho X^2 dS}{D_S(X)}+\frac{\oint\oint_S \rho Y^2 dS}{D_S(Y)}+\frac{\oint\oint_S \rho Z^2 dS}{D_S(Z)} = \frac{3}{8\pi}\oint\oint_S \rho dS = \frac{3M}{8\pi}.$$

It remains to be shown that the conformal mapping taking S to $\hat{S}$ can be chosen such that $\hat{G} = \hat{O}$; this is a consequence of the following lemma:

Lemma 1.1. *Given a density $\widetilde{\rho} \geq 0$ on the sphere $\hat{S}$ (not consisting of point masses), there exists a Möbius transformation $\widetilde{P} \to \hat{P}$ of the sphere $\hat{S}$ to itself, such that $\hat{\rho}(\hat{P}) = \widetilde{\rho}(\widetilde{P})(d\widetilde{S}/d\hat{S})$ has its center of gravity G in the center $\hat{O}$ of the sphere.*

Proof. The proof of the lemma is inspired by the methodology of Szegö [4]. Let N be a point on $\hat{S}$ and k a real number, $0 < k \leq 1$; denote by $H_{N,k}$ the Möbius transformation on S induced by the homothety $\zeta \mapsto k\zeta$ on a complex tangent plane to the sphere at N ($\zeta = 0$ at N): the two fixed points of $H_{N,k}$ are (if $k \neq 1$) N and its antipode; $H_{N,1}$ is the identity. The mass distribution $\widetilde{\rho}$ is mapped by $H_{N,k}$ to $\widetilde{\rho}_{N,k}$, with center of gravity $\widetilde{G}_{N,k}$; $\widetilde{\rho}_{N,1} \equiv \widetilde{\rho}$, $\widetilde{G}_{N,1} = \widetilde{G}$; if $k \searrow 0$, $\widetilde{G}_{N,k} \to N$. The total mass is always conserved: $\widetilde{M}_{N,k} = \widetilde{M} = M$. Let us denote by $\widetilde{G}_k$ the surface formed the points $\widetilde{G}_{N,k}$ where N varies over $\hat{S}$ and k is fixed. (In general, this surface $\widetilde{G}_k$ will intersect itself). When k is close to zero, $\widetilde{G}_k$ contains $\hat{O}$; when $k \nearrow 1$, $\widetilde{G}_k$ is close to the point G. If $\widetilde{G} = \hat{O}$, the desired transformation is the identity; otherwise, $\hat{O}$ is exterior to $\widetilde{G}_k$ for k sufficiently close to 1; thus there exists $0 < \hat{k} < 1$ such that $\widetilde{G}_{\hat{k}} \ni \hat{O}$, $\hat{O} = \widetilde{G}_{\hat{N},\hat{k}} = \hat{G}$, $H_{\hat{N},\hat{k}}$ is the desired Möbius transformation. □

Corollary 1.2.

$$\left(\frac{1}{\mu_2} + 2\frac{1}{\mu_3}\right)\frac{1}{M} \geq \frac{3}{8\pi}$$

and

$$\mu_2 M \leq 8\pi \approx 25.133.$$

□

Example 1. *On a regular tetrahedral surface with $\rho =$ constant,*

$$\mu_2 M = \frac{4\pi^2}{\sqrt{3}} \approx 22.793.$$

2

If J is a Jordan domain; $\rho(P) \geq 0$ a density on J; λ_1 the first eigenvalue of the membrane on J with fixed contour; $\mu_1 = 0 < \mu_2 \leq \mu_3 \leq \dots$ the eigenvalues of the free membrane. Excluding the trivial case of point masses, we have

$$\left(\frac{1}{\lambda_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right)\frac{1}{M} \geq \frac{3}{4\pi} \approx 0.23873. \quad (2)$$

Equality holds in particular when $\rho =$ constant on any hemisphere of radius R (then $\lambda_1 = \mu_2 = \mu_3 = 2/R^2\rho$ and $M = 2\pi R^2 \rho$).

Proof. **A:** One can conformally map J to the hemisphere $\hat{J} : \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 1$, $\hat{z} > 0$, so that the distribution of masses on $\hat{J}$: $\hat{\rho}(\hat{P}) =$

$\rho(P)(dS/d\hat{S})$ has its center of mass $\hat{G}$ on the axis $\hat{O}\hat{z}$. This results from the following lemma:

Lemma 2.1. *Given $\widetilde{\rho} \geq 0$ on the hemisphere $\hat{J}$ (without point masses) there exists a conformal mapping $H_{\hat{N},\hat{k}} : \widetilde{P} \mapsto \hat{P}$ ($\hat{N}$ on the equator E of $\hat{J}$) of $\hat{J}$ onto $\hat{J}$ such that $r\hat{h}o(\hat{P}) = \widetilde{\rho}(\widetilde{P})(d\widetilde{S}/d\hat{S})$ has its center of mass on the axis $\hat{O}\hat{z}$.*

Proof. Again, the proof of this lemma is inspired by Szegö [4]. If $k \searrow 0$, $\widetilde{G}_{N,k} \to N \in E$; that is to say $\widetilde{G}_k$ is the closed curve traced by $\widetilde{G}_{N,k}$ when the point N traverses the equator E. (This curve can intersect itself.) When k is sufficiently small, the coordinate of this curve relative to the $\hat{O}\hat{z}$ axis (or of its horizontal projection relative to $\hat{O}$) is 1; when $k \nearrow 1$, $\widetilde{G}_k \to \widetilde{G}$; if $\widetilde{G}$ is on $\hat{O}\hat{z}$, the desired transformation is the identity; if not, the coordinate is zero for k sufficiently close to 1; since the coordinate moved from 1 to 0, there exists $0 < \hat{k} < 1$ such that $\widetilde{G}_{\hat{k}}$ intersects $\hat{O}\hat{z}$ in a point $\hat{G} = \widetilde{G}_{\hat{N},\hat{k}}$, $H_{\hat{N},\hat{k}}$ is the desired transformation. □

B: Mapping $\hat{x}, \hat{y}, \hat{z}$ from $\hat{J}$ to J, the images $X(P)$, $Y(P)$, $Z(P)$ satisfy

$$\iint_J \rho X dS = \iint_J \rho Y dS = 0, \quad D_J(X,Y) = 0,$$

$$D_J(X) = D_J(Y) = D_J(Z) = \frac{4\pi}{3} \text{ and } X^2 + Y^2 + Z^2 \equiv \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \equiv 1.$$

The function Z is admissible by the principle of Rayleigh characterizing λ_1; the linear space $L(X,Y)$ is admissible for the variational characterization [3] of $\mu_2^{-1} + \mu_3^{-1}$:

$$\frac{1}{\lambda_1} \geq \frac{\iint_J \rho Z^2 dS}{D_J(Z)}; \quad \frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{\iint_J \rho X^2 dS}{D_J(X)} + \frac{\iint_J \rho Y^2 dS}{D_J(Y)};$$

(2) follows.

□

Remark Among plane homogeneous membranes, the circle realizes the maximum of $\lambda_1^{-1} M^{-1}$ [5][1], but the minimum of $\left(\frac{1}{\mu_2} + \frac{1}{\mu_3}\right)\frac{1}{M}$ [4][6]. Here are some examples of the values of the left hand side of (2): circle: 0.24283; square: 0.25330; equilateral triangle: 0.30711; one can conjecture that, among the plane homogeneous membranes of total mass M, the circle realizes the minimum of $\left(\frac{1}{\lambda_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right)\frac{1}{M}$. If true, this property would reinforce the Szegö-Weinberger theorem.

3

If B is a "bilatere" (a Jordan domain with 2 marked boundary points) with "sides" a and b and density given on B by $\rho \geq 0$. Denote by λ_a the first eigenvalue of the membrane fixing a, free on b and by λ_b the first eigenvalue of the membrane fixing b and free on a; and by $\mu_1 = 0 < \mu_2 \leq \dots$ the eigenvalues of the free membrane. Excluding the case of point masses,

$$\left(\frac{1}{\lambda_a} + \frac{1}{\lambda_b} + \frac{1}{\mu_2}\right)\frac{1}{M} \geq \frac{3}{2\pi} \approx 0.47746. \tag{3}$$

We have equality in particular when $\rho =$ constant on a quarter-sphere $\hat{B}$: $\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2$, $\hat{x} > 0$, $\hat{y} > 0$ with $\hat{a}$ and $\hat{b} =$ the two meridian boundaries (then $\lambda_{\hat{a}} = \lambda_{\hat{b}} = \mu_2 = 2/R^2\rho$ and $M = \pi R^2 \rho$).

Proof. A continuity argument (let $\hat{N}$ be the north pole and $\check{S}$ the south pole of $\hat{B}$; if $k \searrow 0$, $\widetilde{G}_{\hat{N},k} \to \hat{N}$; if $k \nearrow \infty$, $\widetilde{G}_{\hat{N},k} \to \check{S}$; then apply the Bolzano theorem) easily shows that one can conformally map B to $\hat{B}$ so that the arc-border a has as its image the meridian $\hat{x} = 0$, the arc b the meridian $\hat{y} = 0$, and the distribution of masses on $\hat{B}$: $\hat{\rho}(\hat{P}) = \rho(P)(dS/d\hat{S})$ has its center of mass $\hat{G}$ in the plane $\hat{z} = 0$. Mapping $\hat{x}, \hat{y}, \hat{z}$ from $\hat{B}$ to B, the images $X(P)$, $Y(P)$, $Z(P)$ satisfy

$$\iint_B \rho z dS = 0, D_B(X) = D_B(Y) = D_B(Z) = \frac{2\pi}{3} \text{ and } X^2+Y^2+Z^2 \equiv \hat{x}^2+\hat{y}^2+\hat{z}^2 \equiv 1;$$

the function X is admissible for the principle of Rayleigh characterizing λ_a, the function Y for λ_b, the function Z for μ_2:

$$\frac{1}{\lambda_a} \geq \frac{\iint_B \rho X^2 dS}{D_B(X)}; \quad \frac{1}{\lambda_b} \geq \frac{\iint_B \rho Y^2 dS}{D_B(Y)}; \quad \frac{1}{\mu_2} \geq \frac{\iint_B \rho Z^2 dS}{D_B(Z)}$$

(3) follows. □

4

Let T be a "trilatere" (Jordan domain with three marked boundary points) with "sides" a, b and c; consider on T a membrane of desnity given by $\rho \geq 0$, fixed in turn along a, b, c; then

$$\left(\frac{1}{\lambda_a} + \frac{1}{\lambda_b} + \frac{1}{\lambda_c}\right)\frac{1}{M} \geq \frac{3}{\pi} \approx 0.95493. \tag{4}$$

Equality holds in particular when $\rho =$ constant on a trirectangular triangle spheric (an eighth of a sphere). This result was already proved ([2]) using a conformal mapping.

For symmetric domaines, (4) implies (3), (3) implies (2) and (2) implies (1). An example of equivalence: on a regular octahedron, $\rho \equiv 1$: the four formulas give the same upper bound $\lambda_1 \leq \frac{4\pi}{\sqrt{3}} \approx 7.2552$ for the first eigenvalue λ_1 of a hexagonal membrane with fixed side length 1.

References

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