Math 8250: Day 9

Now the goal is to show that $(V_r^s)^* \cong M_r^s(V)$, the v.s. of multilinear functions

$$\underbrace{V\times\cdots\times V}_{r}\times\underbrace{V^*\times\cdots\times V^*}_{s}\to\mathbb{R}.$$

To see this, define a pairing $(V^*)_r^s\times V_r^s\to\mathbb{R}$ by

$$(v^*,w)=v_1^*(w_1)\cdots v_{r+s}^*(w_{r+s})$$

where $v^*=v_1^*\otimes\cdots\otimes v_r^*\otimes w_{r+1}\otimes\cdots\otimes w_{r+s}\in(V^*)_r^s$

and $w=w_1\otimes\cdots\otimes w_r\otimes v_{r+1}^*\otimes\cdots\otimes v_{r+s}^*\in V_r^s$.

This pairing is nondegenerate, so we see that $(V^*)_r^s\cong(V_r^s)^*\cong M_r^s(V)$ by the universal property, since any

$$\begin{tikzcd}
V\times\cdots\times V\times V^*\times\cdots\times V^* \arrow[r] \arrow[d] & \mathbb{R} \\
V_r^s = V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^* \arrow[ur, dashed, "\text{lifts to}"'] &
\end{tikzcd}$$

Likewise, we can define a non-deg. pairing $\Lambda^k(V^*)\times\Lambda^k(V)\to\mathbb{R}$ by

$$(v^*,w)=\det\left(\left(v_i^*(w_j)\right)_{i,j}\right)$$

for $v^*=v_1^*\wedge\cdots\wedge v_k^*\in\Lambda^k(V^*)$ and $w=w_1\wedge\cdots\wedge w_k\in\Lambda^k(V)$ and extending linearly.

Then this induces the iso $\Lambda^k(V^*)\cong\Lambda^k(V)^*\cong A_k(V)$.

In turn, $\Lambda(V^*)=\oplus\Lambda^k(V^*)=\oplus\Lambda^k(V)^*=\left(\oplus\Lambda^k(V)\right)^*=\Lambda(V)^*$

In turn, then, since a differential form is a smooth section of $\Lambda^k(M)=\bigcup_p\Lambda^k((T_pM)^*)$, we see that a form is equiv. to an alternating multilinear map $\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to C^\infty(M)$, as claimed.

Digression on Λ^k and Grassmannians

Let $G_k\mathbb{R}^n$ be the set of oriented k-planes in $\mathbb{R}^n$. e.g. $G_1\mathbb{R}^n = \mathbb{RP}^{n-1}$, $G_2\mathbb{R}^3$ = oriented planes in $\mathbb{R}^3 \cong G_1\mathbb{R}^3$

etc. Now, we can define the Plücker embedding $\psi : G_k\mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)/_{\mathbb{R}_+}$ as follows:

for $P \in G_k\mathbb{R}^n$, let $v_1, \dots, v_k$ be a basis for P. Then define $\psi(P) := v_1 \wedge \dots \wedge v_k$.

This is well-defined b/c, if $w_1, \dots, w_k$ is another basis, then $w_i := \sum a_{ij} v_j$ .. Let $A = (a_{ij})$ be the

change-of-basis matrix. Then $w_1 \wedge \dots \wedge w_k = \sum a_{1j} v_j \wedge \sum a_{2j} v_j \wedge \dots \wedge \sum a_{kj} v_j$

$$= \det A \, (v_1 \wedge \dots \wedge v_k)$$

which is the same thing if $\det A > 0$.

Notice: the image of ψ consists precisely of the (non-zero) decomposable k-vectors, meaning those elts of $\Lambda^k(\mathbb{R}^n)$

that can be written as $v_1 \wedge \dots \wedge v_k$ for some $v_1, \dots, v_k$.

Now: focus on $G_2\mathbb{R}^4$. I want to use ψ to say exactly what $G_2\mathbb{R}^4$ is.

From the above, can identify $G_2\mathbb{R}^4$ w/ decomposable, non-zero elts of $\Lambda^2\mathbb{R}^4$ mod scaling

So, scaling to length 1, some subset of the unit sphere in $\Lambda^2\mathbb{R}^4 \cong \mathbb{R}^6$.

If $e_1, \dots, e_4$ is an ON basis for $\mathbb{R}^4$, then $e_1\wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4$ gives an ON basis

for $\Lambda^2\mathbb{R}^4$ (if you like, this is the definition of the inner product on $\Lambda^2\mathbb{R}^4$).

So the goal is to identify the unit length, decomposable elts of $\Lambda^2\mathbb{R}^4$.

<u>Prop:</u> $\omega \in \Lambda^2\mathbb{R}^4$ is decomposable $\iff \omega \wedge \omega = 0$.

<u>Proof:</u> Exercise

Define the endomorphism $\star$ of $\Lambda^2\mathbb{R}^4$ by

$$e_1\wedge e_2 \overset{\star}{\longleftrightarrow} e_3\wedge e_4$$
$$e_1\wedge e_3 \overset{\star}{\longleftrightarrow} -e_2\wedge e_4$$
$$e_1\wedge e_4 \overset{\star}{\longleftrightarrow} e_2\wedge e_3$$

and extend linearly.

Note: $\star\psi(P) = \psi(P^\perp)$, where $P^\perp$ is the orthogonal complement of P.

Prop: $\omega \in \Lambda^2\mathbb{R}^4$ is decomposable $\Leftrightarrow$ ω is perpendicular to $\star\omega$.

Proof: If $\omega = \sum a_{ij} e_i\wedge e_j$, then $\langle \omega, \star\omega\rangle = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$.

OTOH, $\omega\wedge\omega = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\, e_1\wedge e_2\wedge e_3\wedge e_4$, so $\langle\omega,\star\omega\rangle = 0 \Leftrightarrow \omega\wedge\omega = 0$,

which happens $\Leftrightarrow$ ω is decomposable. $\blacksquare$

Now, $\star$ is an isometric (length-preserving) involution, so the only possible eigenvalues are ± 1.

Let E_+ be the $+1$-eigenspace & E_- be the -1-eigenspace. Then can check that, for any

$\omega \in \Lambda^2\mathbb{R}^4$, $\omega = \frac{\omega+\star\omega}{2} + \frac{\omega - \star\omega}{2}$ with $\frac{\omega+\star\omega}{2} \in E_+$ and $\frac{\omega-\star\omega}{2} \in E_-$.

Can write bases $\frac{e_1\wedge e_2 \pm e_3\wedge e_4}{2}, \frac{e_1\wedge e_3 \mp e_2\wedge e_4}{2}, \frac{e_1\wedge e_4 \pm e_2\wedge e_3}{2}$ for $E_\pm$, each of which is 3-dim'l.

Prop: $\omega\in\Lambda^2\mathbb{R}^4$ is decomposable $\Leftrightarrow$ $\frac{\omega+\star\omega}{2}$ & $\frac{\omega-\star\omega}{2}$ have the same length.

Proof: ω is decomposable $\Leftrightarrow$ $\langle\omega,\star\omega\rangle = 0$ which happens $\Leftrightarrow$

$$0 = \langle \omega, \star\omega\rangle = \left\langle \frac{\omega+\star\omega}{2} + \frac{\omega-\star\omega}{2}, \frac{\omega+\star\omega}{2} - \frac{\omega-\star\omega}{2}\right\rangle = \left\|\frac{\omega+\star\omega}{2}\right\|^2 - \left\|\frac{\omega-\star\omega}{2}\right\|^2 \qquad \blacksquare$$

Cor: Since $\frac{\omega+\star\omega}{2}$ & $\frac{\omega-\star\omega}{2}$ are perpendicular, ω is a unit decomposable 2-vector $\Leftrightarrow$ both have length $\frac{1}{\sqrt{2}}$.

Therefore, if $S^2_+(1/\sqrt{2})$ is the $\frac{1}{\sqrt{2}}$-sphere in E_+ & $S^2_-(1/\sqrt{2})$ is the $\frac{1}{\sqrt{2}}$-sphere in E_-, we can identify $G_2\mathbb{R}^4$ w/ $S^2_+(1/\sqrt{2}) \times S^2_-(1/\sqrt{2})$ (in fact, this is an isometry).