Math 8250: Day 8

Last time, I think I screwed up the definition of a tensor field. Should have been

Def: A *tensor field* of type (r,s) on M is a smooth section of the (r,s)-tensor bundle which is to say a smooth map $\rho : M \to T^s_r(M)$ so that $\rho(p) \in (T_pM)^s_r$.

Ex: Using the earlier example, a Riemannian metric g on M is a symmetric $(0,2)$-tensor field on M. More prosaically, a vector field is a $(1,0)$-tensor field on M.

Differential forms will be sort of like tensor fields, but we need to take an appropriate quotient of the tensor algebra to get the *exterior algebra*.

Specifically, if $C(V) := \bigoplus_k V^0_k$ and $\mathcal{I}(V)$ is the two-sided ideal generated by elements of the form $v \otimes v$ for $v \in V$, then we define the *exterior algebra* of V to be the quotient

$$\Lambda(V) := C(V)/\mathcal{I}(V).$$

This will be a graded algebra w/ product denoted by $\wedge$. Of course, the product is just the product induced by $\otimes$: the residue class of $v_1 \otimes \cdots \otimes v_k$ in $\Lambda(V)$ is denoted $v_1 \wedge \cdots \wedge v_k$.

The grading is just given by the degree of the wedge product, since

$$\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V),$$

where $\Lambda^k(V) = V^0_k / \mathcal{I}_k$ and $\mathcal{I}_k = \mathcal{I}(V) \cap V_{k,0}$.

Prop: If $v \in \Lambda^k(V)$, $w \in \Lambda^l(V)$, then $w \wedge v = (-1)^{kl} v \wedge w$

Proof: Exercise

Cor: $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = (-1)^{\operatorname{sgn}(\sigma)} v_1 \wedge \cdots \wedge v_k$ for $\sigma \in S_k$.

Prop: If $\{e_1,\dots,e_n\}$ is a basis for V, then $\{e_{i_1}\wedge\dots\wedge e_{i_k} \mid 1\le i_1<i_2<\dots<i_k\le n\}$ is a basis for $\Lambda^k(V)$. In particular,

$$\dim \Lambda^k(V)=\binom{n}{k}$$ for $0\le k\le n$ and zero otherwise, so $\dim \Lambda(V)=2^n$.

Proof: Exercise.

The exterior algebra also satisfies a universal property:

Thm (Universal Mapping Property): If $\varphi: V\times\dots\times V\to\Lambda^k(V)$ is given by $(v_1,\dots,v_k)\mapsto v_1\wedge\dots\wedge v_k$ and if

$F: V\times\dots\times V\to U$ is an alternating multilinear map, then $\exists!$ linear map $\tilde{F}$ s.t. the diagram commutes:

$$\begin{tikzcd}
V\times\dots\times V \arrow[r, "F"] \arrow[d, "\varphi"'] & U \\
\Lambda^k(V) \arrow[ur, dashed, "\tilde{F}"'] &
\end{tikzcd}$$

So the exterior algebra is a gadget for turning alternating, multilinear maps into linear maps.

Notice, when $U=\mathbb{R}$, we have that $\Lambda^k(V)^*\cong A_k(V)$, the space of alternating, multilinear functions on $\underbrace{V\times\dots\times V}_{k}$.

Now, we can define the exterior k bundle over M

$$\Lambda^k(M)=\bigcup_{p\in M}\Lambda^k((T_pM)^*)$$

and the exterior algebra bundle over M

$$\Lambda(M):=\bigcup_{p\in M}\Lambda((T_pM)^*)$$

Then a differential k-form on M is a smooth section of $\Lambda^k(M)$; i.e. a smooth map $\omega: M\to\Lambda^k(M)$ s.t.

$\omega(p)\in\Lambda^k((T_pM)^*)$.

Earlier I claimed that k-forms were alternating multilinear maps $\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)\to C^\infty(M)$ ... what gives?
Well, in general it turns out that $\Lambda^k(V^*)\cong A_k(V)$, which implies the above w/ $V=T_pM$.

To see this, recall that a non-degenerate pairing of vector spaces V & W is a bilinear map (i.e. bilinear form)

$$(\;,\;): V\times W \to \mathbb{R}$$

s.t. $W \ni w \neq 0$ implies $(v,w)\neq 0$ for some $v\neq 0$ (& the other way).

Equivalently, a pairing is a linear map $V\otimes W\to\mathbb{R}$ (i.e. an elt. of $(V\otimes W)^*$).

Clearly, if V & W are non-degenerately paired, then $\varphi: V\to W^*$, $\varphi(v)(w)=(v,w)$ and $\psi: W\to V^*$, $\psi(w)(v)=(w,v)$

are both injective, so φ & ψ are isomorphisms (of course, any inj is iso. for finite dim'l vector spaces)

Ex: An inner product on V is a non-degenerate pairing of V w/ itself, so gives an iso. $V\to V^*$.

Now the goal is to show that $(V_r^s)^* \cong M_r^s(V)$, the v.s. of multilinear functions

$$\underbrace{V\times\cdots\times V}_{r}\times\underbrace{V^*\times\cdots\times V^*}_{s}\to\mathbb{R}.$$

To see this, define a pairing $(V^*)_r^s\times V_r^s\to\mathbb{R}$ by

$$(v^*,w)=v_1^*(w_1)\cdots v_{r+s}^*(w_{r+s})$$

where $v^*=v_1^*\otimes\cdots\otimes v_r^*\otimes w_{r+1}\otimes\cdots\otimes w_{r+s}\in(V^*)_r^s$

and $w=w_1\otimes\cdots\otimes w_r\otimes v_{r+1}^*\otimes\cdots\otimes v_{r+s}^*\in V_r^s$.

This pairing is nondegenerate, so we see that $(V^*)_r^s\cong(V_r^s)^*\cong M_r^s(V)$ by the universal property, since any

$$\begin{tikzcd}
V\times\cdots\times V\times V^*\times\cdots\times V^* \arrow[r] \arrow[d] & \mathbb{R} \\
V_r^s = V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^* \arrow[ur, dashed, "\text{lifts to}"'] &
\end{tikzcd}$$