Math 8250: Day 7

Recall the following diagram from last time:

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M) \\
C^\infty(M) \arrow[r, "\nabla"] \arrow[u, equal] & \mathfrak{X}(M) \arrow[r, "\nabla\times"] \arrow[u, "\flat", "\cong"'] & \mathfrak{X}(M) \arrow[r, "\nabla\cdot"] \arrow[u, "*\flat", "\cong"'] & C^\infty(M) \arrow[u, "\cong"']
\end{tikzcd}$$

I haven't really said what d is, but the exterior derivative of a 0-form corresponds to the gradient, the exterior derivative of an $(n-1)$-form corresponds to the divergence, and in the special case of a 3-mfd, the exterior derivative of a 1-form corresponds to the curl.

Hopefully the fact that in this language div, grad, & curl are all really the same thing is a good advertisement for differential forms (and, in fact, with this in hand, the fundamental theorem of calculus, Green's theorem, and the divergence theorem are all really the same theorem: Stokes' Theorem)

We will be spending quite a bit of time getting a handle on what all these things are &, in particular, on what the image & kernel of d look like.

Notice 2 more things from the diagram:

① Since $\nabla\times\nabla f=0$, $\nabla\cdot\nabla\times X=0$ for all $f\in C^\infty(M)$ and $X\in\mathfrak{X}(M)$, we have that $d\circ d=0$ whether we start at 0-forms or 1-forms. In other words, the sequence $\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\Omega^2(M)\xrightarrow{d}\Omega^3(M)$ is a chain complex.

In fact, the exterior derivative will always give rise to a chain complex

$$\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{n-1}(M)\xrightarrow{d}\Omega^n(M),$$

meaning that $d\circ d=0$ at every stage.

As always w/ a chain complex, we will take the homology; i.e., if $d_k:\Omega^k(M)\to\Omega^{k+1}(M)$

then we will define the k^{th} de Rham cohomology group to be

$$H^k_{dR}(M) := \ker d_k / \operatorname{im} d_{k-1}$$

In the case $n=3, k=1$, this gives us a v.s. isomorphic to the space of curl-free fields which are not gradients. Of course, the question one could ask is whether we can do better than just "a v.s. isomorphic to"... and the answer, in the form of the Hodge Theorem, will be "yes".

Now, de Rham's Theorem says that, as vector spaces,

$$H^k_{dR}(M) \cong H^k(M;\mathbb{R}),$$

the k^{th} singular cohomology group with real coefficients. So we capture the topology of M in this way.

② Since $\Omega^1(M) \simeq \mathfrak{X}(M)$ and $\Omega^2(M) \simeq \mathfrak{X}(M)$, we have that $\Omega^1(M) \simeq \Omega^2(M)$.

Likewise, $\Omega^0(M) = C^\infty(M)$ and $\Omega^3(M) \simeq C^\infty(M)$, so $\Omega^0(M) \simeq \Omega^3(M)$.

In general, on an n-manifold M, it will turn out that $\Omega^k(M) \simeq \Omega^{n-k}(M)$ for all $0 \leq k \leq n$.

This is a souped-up form of Poincaré duality, and a consequence will be the following version of Poincaré duality:

$$H^k_{dR}(M) \cong H^{n-k}_{dR}(M) \Rightarrow H^k(M;\mathbb{R}) \cong H^{n-k}(M;\mathbb{R}).$$

Tensor Algebras & Exterior Algebras

Given vector fields V & W, we define the tensor product $V \otimes W$ to be the vector space whose elements are $v \otimes w$ for $v \in V, w \in W$ s.t.

① $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

② $v\otimes(w_1+w_2)=v\otimes w_1+v\otimes w_2$

③ $a(v\otimes w)=(av)\otimes w=v\otimes(aw)$

Ex: If $V=\mathbb{R}^m$, $W=\mathbb{R}^n$, then $v\otimes w$ corresponds to the $m\times n$ matrix $vw^T=\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_m\end{bmatrix}[w_1\cdots w_n]$.

Thm (Universal Mapping Property): If $\varphi: V\times W\to V\otimes W$ is the map given by $(v,w)\mapsto v\otimes w$ and $F: V\times W\to U$ is a bilinear map, then $\exists!$ linear map $\tilde{F}$ making the diagram commute:

$$\begin{tikzcd} V\times W \arrow[r, "F"] \arrow[d, "\varphi"'] & U \\ V\otimes W \arrow[ur, dotted, "\tilde{F}"'] & \end{tikzcd}$$

Moreover, $V\otimes W$ is unique in this regard: any other v.s. w/ this property is isomorphic to $V\otimes W$.

So this is a way of turning bilinear (or multilinear) maps into linear maps.

Prop: If $e_1,\dots,e_m$ is a basis for V and $f_1,\dots,f_n$ is a basis for W, then

$\{e_i\otimes f_j\}$ is a basis for $V\otimes W$.

In particular, $\dim V\otimes W = mn$.

Def: The tensor space of type (r,s) associated with V is the vector space

$$V_r^s := \underbrace{V\otimes\cdots\otimes V}_{r}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{s}$$

Ex: An inner product $\langle\,,\rangle$ on V is a symmetric $(0,2)$-tensor on V:

$\langle\,,\rangle$ defines a bilinear map $V\times V\to\mathbb{R}$. By the universal property, this induces a linear map

$V\otimes V\to\mathbb{R}$.

So we uniquely identify $\langle\,,\rangle$ with an element of $(V\otimes V)^* = V^*\otimes V^*$ to be shown

The direct sum $T(V) := \bigoplus_{r,s\geq 0} V_r^s$ where $V_0^0 =$ ground field of V is the tensor algebra of V.

Notice that $T(V)$ is noncommutative ($e_1 \otimes e_2 \neq e_2 \otimes e_1$), associative, and (bi-)graded.

We will define the tensor bundle of type (r,s) over M to be

$$T_r^s(M) := \bigcup_{p\in M} (T_pM)_r^s,$$

which is a smooth mfd, and a tensor field of type (r,s) on M is a smooth section of the (r,s)-tensor bundle which is to say a smooth map $M \to T_r^s(M)$.

Ex: Using the example above, a Riemannian metric g on M is a symmetric $(0,2)$-tensor field on M.
More prosaically, a vector field is a $(1,0)$-tensor field on M.