Math 8250: Day 6

Differential Forms

"Def": A smooth differential k-form on a mfd M is a C^∞, alternating, multilinear map

$$\omega: \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k \text{ copies}} \to C^\infty(M)$$

where $\mathfrak{X}(M)$ is the $C^\infty(M)$ module of vector fields on M.

(i.e. $\omega(X_1, \ldots, X_{i-1}, fX + gY, X_{i+1}, \ldots, X_k) = f\omega(X_1, \ldots, X_{i-1}, X, X_{i+1}, \ldots, X_n) + g\omega(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n)$

& $\omega(X_1, \ldots, -X_i, \ldots, X_n) = -\omega(X_1, \ldots, X_i, \ldots, X_n)$)

The vector space of k-forms on M is denoted $\Omega^k(M)$.

That def. is kind of vague & useless, so let's see a somewhat more concrete example:

Ex: Let M be a mfd & let $X \in \mathfrak{X}(M)$ be a vector field. Assume M has a Riemannian metric g, meaning that for any $p \in M$ and $u, v \in T_p(M)$, $g_p(u,v)$ defines an inner product on T_pM.

Then X defines a 1-form α as follows:

$$\alpha(Y)(p) = g_p(X, Y).$$

The map $\alpha: \mathfrak{X}(M) \to C^\infty(M)$ is definitely smooth.

Alternating: $\alpha(-Y)(p) = g_p(X, -Y) = -g_p(X,Y) = -\alpha(Y)(p)$

Linear: $\alpha(fY + hZ)(p) = g_p(X, f(p)Y + h(p)Z) = f(p)g_p(X,Y) + h(p)g_p(X,Z) = f(p)\alpha(Y)(p) + h(p)\alpha(Z)(p)$

So satisfies the conditions. OTOH, any differential 1-form $\beta: \mathfrak{X}(M) \to C^\infty(M)$ is just a linear functional on each $T_p(M)$, so $\beta_p \in (T_pM)^*$. But then $\exists\, u_p \in T_pM$ s.t. $\beta_p(v) = g_p(u_p, v)$ for all $v \in T_p(M)$ (if you like, this is the Riesz representation theorem). The smoothness of β allows us to form the global vector field $Y \in \mathfrak{X}(M)$ where $Y(p) = u_p$ for all $p \in M$.

In other words, a Riemannian metric gives us an isomorphism b/w $\mathfrak{X}(M)$ & $\Omega^1(M)$.

Also, a 0-form is clearly just a C^∞ function on M, so $\Omega^0(M) = C^\infty(M)$.

<u>2 Questions:</u> ① What about other k?

② What are the gradient & divergence (and curl in 3 dimensions) in this language?

The first part of ② is easiest, so let's focus on that for a second.

The gradient of a function is a vector field, given in local coords by

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

So the corresponding 1-form is the form α s.t.

$$\alpha(Y)(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} b_i = \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i} = \left(\sum b_i \frac{\partial}{\partial x_i}\right) f = Y(f)(p) = df_p(Y)$$

In other words, the 1-form identified with ∇f is exactly the differential of f, df.

(In general, note that differentials are 1-forms: given $f \in C^\infty(M)$, $df_p \in (T_pM)^*$, so $df \in T^*M \cong \Omega^1(M)$.)

i.e.
$$\begin{array}{ccc} \nabla : C^\infty(M) & \longrightarrow & \mathfrak{X}(M) \\ \| & & \wr\| \\ d : \Omega^0(M) & \longrightarrow & \Omega^1(M) \\ f & \longmapsto & df \end{array}$$
corresponds to the exterior derivative

Later we will formulate an invariant (i.e. coordinate independent) definition of the exterior derivative, but first let's say a little more about ① & ②.

It will turn out that the divergence also corresponds to an exterior derivative, this time $d : \Omega^{n-1}(M) \to \Omega^n(M)$, where $\Omega^{n-1}(M) \cong \mathfrak{X}(M)$ and $\Omega^n(M) \cong C^\infty(M)$.

In general, an n-form on M^n is an alternating multilinear map $(\mathfrak{X}(M))^n \to C^\infty(M)$. At each $p \in M$, $T_pM \cong \mathbb{R}^n$, so ω_p is an alternating multilinear map $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n} \to \mathbb{R}$. Up to a scalar, the unique such is the determinant, so each ω_p is a scalar mult. of the determinant. Therefore, $\omega = f \, dvol_M$, where $dvol_M$ is the name we give to the n-form which is just the determinant on each T_pM. Therefore, $\Omega^n(M^n) \cong C^\infty(M)$.

Let's make this all more explicit on 3-manifolds

Ex: Let M^3 be a smooth 3-manifold. As discussed above, a vector field $X \in \mathfrak{X}(M)$ can be identified with a 1-form

$$\alpha = X^\flat$$

via the so-called musical isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ (the inverse is $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$)

The cool thing about 3-manifolds is that we have a cross product, which allows us to define the vector triple product. As a result, we can also identify a vector field X w/ a 2-form $\omega = \star(X^\flat)$ as follows:

$$\star(X^\flat)_p(Y,Z) = g_p(X, Y \times Z) = \det[X \ Y \ Z]$$

if X, Y, Z are written in orthonormal (w.r.t. g) local coordinates.

Check that this is a 2-form:

- Smooth ✓
- Alternating: $\star(X^\flat)_p(-Y,Z) = g_p(X, (-Y)\times Z) = g_p(X, -(Y\times Z)) = -g_p(X, Y\times Z) = -\star(X^\flat)_p(Y,Z)$ ✓
- Multilinear ✓

(Really, don't need to do anything for the last 2, since det is alternating & multilinear)

So this gives us an identification $\Omega^2(M) \cong \mathfrak{X}(M)$

Finally, we saw that $\Omega^3(M) \cong C^\infty(M)$. Now we have the following diagram:

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] \arrow[d, equal] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M) \\
C^\infty(M) \arrow[r, "\nabla"] & \mathfrak{X}(M) \arrow[r, "\nabla\times"] \arrow[u, "\flat", "\cong"'] & \mathfrak{X}(M) \arrow[r, "\nabla\cdot"] \arrow[u, "\star\flat", "\cong"'] & C^\infty(M) \arrow[u, "\cong"']
\end{tikzcd}$$

I haven't really said what d is, but in the exterior derivative of a 0-form corresponds to the gradient, the exterior derivative of an $(n-1)$-form corresponds to the divergence, and in the special case of a 3-mfd, the exterior derivative of a 1-form corresponds to the curl.

Hopefully the fact that in this language div, grad, & curl are all really the same thing is a good advertisement for differential forms (and, in fact, with this in hand, the fundamental theorem of calculus, Green's theorem, and the divergence theorem are all really the same theorem: Stokes' Theorem)

We will be spending quite a bit of time getting a handle on what all these things are &, in particular, on what the image & kernel of d look like.

Notice 2 more things from the diagram:

① Since $\nabla \times \nabla f = 0$, $\nabla \cdot \nabla \times X = 0$ for all $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that $d \circ d = 0$ whether we start at 0-forms or 1-forms. In other words, the sequence $\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M)$ is a chain complex.

In fact, the exterior derivative will <u>always</u> give rise to a chain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M),$$

meaning that $d \circ d = 0$ at every stage.

② Since $\Omega^1(M) \cong \mathfrak{X}(M)$ and $\Omega^2(M) \cong \mathfrak{X}(M)$, we have that $\Omega^1(M) \cong \Omega^2(M)$.

Likewise, $\Omega^0(M) = C^\infty(M)$ and $\Omega^3(M) \cong C^\infty(M)$, so $\Omega^0(M) \cong \Omega^3(M)$.

In general, on an n-manifold M, it will turn out that $\Omega^k(M) \cong \Omega^{n-k}(M)$ for all $0 \le k \le n$.