Math 8250: Day 5

Ex: Consider $M = O(n)$. Then an element of $T_I O(n)$ is a tangent vector; i.e., the tangent to a curve through I.

Let $\alpha : (-\varepsilon, \varepsilon) \to O(n)$ s.t. $\alpha(0) = I$. Then $\alpha(t)$ is an orthogonal matrix for all $t \in (-\varepsilon, \varepsilon)$, so

$$\alpha(t)\alpha(t)^T = I$$

Differentiate: $\alpha'(t)\alpha(t)^T + \alpha(t)\alpha'(t)^T = 0$.

At $t=0$, this reduces to $\alpha'(t) + \alpha'(t)^T = 0$, so $\alpha'(t)$ is skew-symmetric.

In other words, $T_I O(n) =$ skew-symmetric $n \times n$ matrices.

Now, if $Q \in O(n)$ and we define $l_Q : O(n) \to O(n)$ by $l_Q(A) = QA$, then

$$(dl_Q)_I \Delta = Q\Delta.$$

Since $(dl_Q)_I : T_I O(n) \to T_Q O(n)$ is clearly surjective, we see that $T_Q O(n)$ consists of matrices of the form $Q\Delta$ where Δ is skew-symmetric.

Therefore, a vector field X on M is given at each point $Q \in M$ as $X_Q = Q\Delta_Q$.

A vector field where $\Delta_Q = \Delta_I \ \forall Q \in M$ is called *left-invariant*.

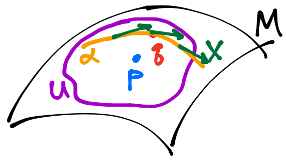
If X & Y are left-invt & $X_I = \Delta_1$, $Y_I = \Delta_2$, then $[X,Y]_I = \Delta_1\Delta_2 - \Delta_2\Delta_1$ &

$$[X,Y]_Q = Q(\Delta_1\Delta_2 - \Delta_2\Delta_1).$$

Def: Let X be a vector field on M. A curve $\alpha : (a,b) \to M$ is called an *integral curve* (or *trajectory*) of X if $\alpha'(t) = X(\alpha(t))$ for all $t \in (a,b)$.

Prop: Let X be a v.f. on M and let $p \in M$. Then there exist a nbhd U of p, an interval $(-\delta, \delta)$, $\delta > 0$ and a map $\varphi : (-\delta,\delta) \times U \to M$ s.t. $t \mapsto \varphi(t,q)$ is the unique smooth curve satisfying

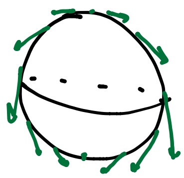
$$\frac{\partial \varphi}{\partial t} = X(\varphi(t,q)) \quad \text{and} \quad \varphi(0,q) = q.$$



Proof: Since M is locally diffeomorphic w/ $\mathbb{R}^n$, this follows from the fundamental theorem on existence & uniqueness of ODEs. $\boxed{}$

Often denote $\varphi_t(q) = \varphi(t,q)$, called the **local flow** of X.

Ex:



$M = S^2$ $\quad X = (zx, zy, z^2-1)$ Then the local flow pushes the mass of the sphere down towards the south pole

> Note: This is the image under diff'l of inverse stereo. proj. of the vector field $(-x,-y)$ on $\mathbb{R}^2$

This is an example of a so-called **complete** vector field b/c the flow exists for all time at every point.

If the manifold were instead the sphere w/ the antarctic circle removed, then the vector field would not be complete.

Now we can interpret the bracket $[X,Y]$ as a derivation of Y along the trajectories of X.

Prop: Let X and Y be smooth vector fields on M, let $p \in M$, and let φ_t be the local flow of X in a nbhd U of p. Then

$$[X,Y](p) = \lim_{t\to 0} \frac{1}{t}[Y - d\varphi_t Y](\varphi_t(p))$$

Proof: For f differentiable in a nbhd of p, our goal is to show that

$$\lim_{t\to 0} \frac{1}{t}((Y - d\varphi_t Y)f)(\varphi_t(p)) = ((XY-YX)f)(p).$$

Notice that $((d\varphi_t Y)f)(\varphi_t(p)) = Y(f\circ\varphi_t)(p)$, so LHS is

$$\lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - Y(f\circ\varphi_t)(p)}{t}$$

If we could get $\lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - (Yf)(p)}{t} = \frac{d(Yf\circ\varphi_t)}{dt}\Big|_{t=0} = X(Yf)(p)$, we would be in business.

To that end, define $h(t,q) := f(\varphi_t(q)) - f(q)$ for all q in the domain of f and t sufficiently small.

Claim: $\exists$ smooth function g s.t. $t\,g(t,q) = h(t,q)$; in particular, $g(0,q) = \frac{\partial h(t,q)}{\partial t}\Big|_{t=0}$.

Assuming the claim, we have that

$$(f\circ\varphi_t)(q) = f(q) + t\,g(t,q) \quad \text{and} \quad g(0,q) = Xf(q).$$

But then

$$((d\varphi_t Y) f)(\varphi_t(p)) = (Y(f\circ\varphi_t))(p) = (Yf)(p) + t\,Yg(t,p)$$

and

$$\lim_{t\to 0} \frac{1}{t}((Y - d\varphi_t Y) f)(\varphi_t(p)) = \lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - Yf(p)}{t} - Yg(0,p)$$

$$= (X(Yf))(p) - (Y(Xf))(p)$$

$$= ([X,Y] f)(p).$$

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Proof of Claim: Define $g(t,q) := \int_0^1 \frac{\partial h(ts,q)}{\partial (ts)}\,ds$

Then, letting $u = ts$, $du = t\,ds$, so

$$g(t,q) = \frac{1}{t}\int_0^1 \frac{\partial h(ts,q)}{\partial (ts)}\,t\,ds = \frac{1}{t}\int_0^t \frac{h(u,q)}{\partial u}\,du = \frac{1}{t}h(t,q),$$

as desired.

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