Math 8250: Day 40

Remember the Hodge-Morrey-Friedrichs decomposition:

$$\Omega^p(M) = c\mathcal{E}^p_N(M) \oplus \mathcal{H}^p(M) \oplus \mathcal{E}^p_D(M)$$

and

$$\mathcal{H}^p(M) = \mathcal{H}^p_N(M) \oplus \mathcal{E}\mathcal{H}^p(M)$$
$$= c\mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}^p_D(M)$$

where
$$\mathcal{H}^p_N(M) \cong H^p(M;\mathbb{R})$$
$$\mathcal{H}^p_D(M) \cong H^p(M,\partial M;\mathbb{R})$$

Now in general $\mathcal{H}^p_N(M)$ & $\mathcal{H}^p_D(M)$ are not orthogl. However, when all of the cohomology of M comes from the bdry (e.g. when M^n is a compact subdomain of $\mathbb{R}^n$), then we get the following 5-term decomp.

Prop: If M^n is a cpct Riem. mfd w/ all cohomology coming from the bdry, then

$$\Omega^p(M) = c\mathcal{E}^p_N(M) \oplus \mathcal{H}^p_N(M) \oplus \mathcal{E}c\mathcal{E}^p(M) \oplus \mathcal{H}^p_D(M) \oplus \mathcal{E}^p_D(M)$$

where $\quad \mathcal{E}c\mathcal{E}^p(M) = \{\omega \in \Omega^p(M) : \omega = d\alpha \text{ for } \alpha \in \Omega^{p-1}(M) \text{ \& } \omega = \delta\beta \text{ for } \beta \in \Omega^{p+1}(M)\}$

This is an orthogonal direct sum decomposition.

Now, let's specialize to compact subdomains in $\mathbb{R}^3$. In this case all cohomology comes from the bdry, so the above applies. Moreover, we have the duality b/w forms & vector fields:

$$\begin{tikzcd}
C^\infty(M) \arrow[r, shift left, "\nabla"] & \mathfrak{X}(M) \arrow[l, shift left, "\nabla\cdot"] \arrow[r, shift left, "\nabla\times"] & \mathfrak{X}(M) \arrow[l, shift left, "\nabla\times"] \arrow[r, shift left, "\nabla\cdot"] & C^\infty(M) \arrow[l, shift left, "\nabla"] \\
\Omega^0(M) \arrow[u, equal] \arrow[r, shift left, "d"] & \Omega^1(M) \arrow[u, "\cong"] \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Omega^2(M) \arrow[u, "\cong"] \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Omega^3(M) \arrow[u, "\cong"] \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

Letting $p=1$ & reinterpreting the above decomposition in terms of vector fields then yields

$$\begin{array}{ccccccccccc}
\mathfrak{X}(M) = & FK & \oplus & HK & \oplus & CG & \oplus & HG & \oplus & GG \\
\uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^p(M) = & c\mathcal{E}^p_N(M) & \oplus & \mathcal{H}^p_N(M) & \oplus & \mathcal{E}c\mathcal{E}^p(M) & \oplus & \mathcal{H}^p_D(M) & \oplus & \mathcal{E}^p_D(M)
\end{array}$$

where

$FK =$ "fluxless knots" $= \{\nabla\cdot V = 0,\ V\cdot\vec{n} = 0,$ all interior fluxes are $0\} = (\ker(\nabla\times))^\perp$

$HK =$ "harmonic knots" $= \{\nabla\cdot V = 0,\ \nabla\times V = 0,\ V\cdot\vec{n} = 0\} = (\ker(\nabla\times)) \cap (\mathrm{im}(\nabla))^\perp$

$CG =$ "curly gradients" $= \{V = \nabla\varphi,\ \nabla\cdot V = 0,$ all boundary fluxes are $0\} = (\mathrm{im}(\nabla)) \cap (\mathrm{im}(\nabla\times))$

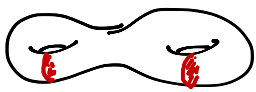
$HG =$ "harmonic gradients" $= \{V = \nabla\varphi,\ \nabla\cdot V = 0,\ \varphi$ locally constant on $\partial M\} = (\ker(\nabla\cdot)) \cap (\mathrm{im}(\nabla\times))^\perp$

$GG =$ "grounded gradients" $= \{V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0\} = (\ker(\nabla\cdot))^\perp$

Moreover, $HK \cong H_1(M;\mathbb{R}) \cong H_2(M,\partial M;\mathbb{R}) = \mathbb{R}^{\text{genus of } \partial M}$

$HG \cong H_2(M;\mathbb{R}) \cong H_1(M,\partial M;\mathbb{R}) = \mathbb{R}^{|\pi_0(\partial M)| - |\pi_0(M)|}$

An interior flux means a flux through a cross-sectional surface, like these disks:

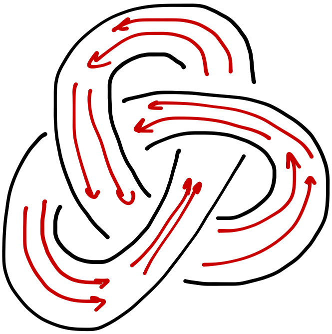


Such cross-sectional disks generate $H_2(M,\partial M;\mathbb{R})$, so by Poincaré duality the vector of fluxes through such disks determines the class in $H_1(M;\mathbb{R})$ represented by the vector field. In particular, the "all interior fluxes zero" condition ensures that vector fields in FK don't represent in cohomology.

A boundary flux means the flux of a v.f. through a boundary component. Since all cohomology comes from the boundary, the bdry components generate $H_2(M;\mathbb{R})$, & so the vector of fluxes of a v.f. through the bdry components determines the class in $H_1(M,\partial M;\mathbb{R})$ represented by a v.f.

In particular, the "all boundary fluxes zero" condition ensures that elts of GG & CG don't represent in cohomology.

Note: Let $K = \text{"knots"} = \{\nabla\cdot V = 0,\ V\cdot\vec{n} = 0\}$. Then K represents an incompressible fluid flow.

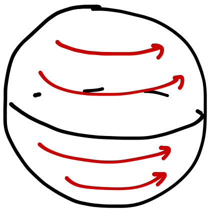


Think of thickening a knot & then letting a fluid flow through a tube centered on the knot.

Examples of vector fields from the 5 different spaces:

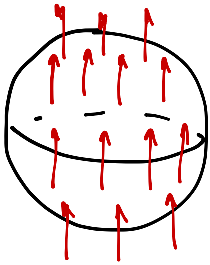
Since B^3 is contractible, $H_1(B^3;\mathbb{R}) = H_2(B^3;\mathbb{R}) = 0$, so

$$\mathcal{X}(B^3) = FK \oplus CG \oplus GG$$



FK: $V = -y\vec{i} + x\vec{j}$

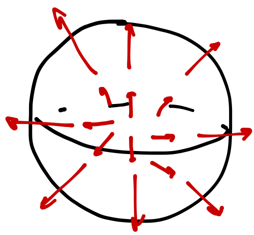
Check $\nabla\cdot V = 0$, $V\cdot\vec{n} = 0$



CG: $V = \vec{k}$

Check $\nabla\cdot V = 0$, bdry flux $= 0$,

$V = \nabla z$



GG: Let $\varphi(x,y,z) = r^2 - 1 = x^2+y^2+z^2-1$. Then $\varphi|_{\partial B^3} = 0$.

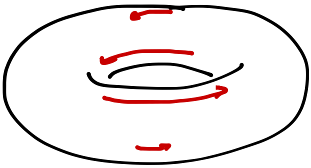
Let $V = \nabla\varphi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

To get harmonic knots & harmc gradients, we need domains w/ topology.

If M = solid torus, then $H_1(M;\mathbb{R}) \cong \mathbb{R}$, so should get a harmc knot. Indeed, if we let (r, φ, z) be cylindrical coords & assume M is a torus of revolution centered on the z-axis, then

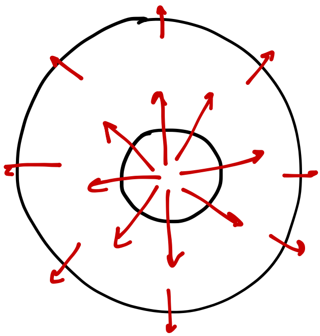
<u>HK:</u> $V = \frac{1}{r}\frac{\partial}{\partial \varphi}$

is divergence-free, curl-free, & tangent to the bdry of M (in fact, it's the magnetic field given by running a steady current along the z-axis



Since $H_2(M;\mathbb{R}) = \{0\}$, we need to switch domains to get a harmc gradient.

So now let M be the region b/w concentric spheres of radii 1 & 2. Then $H_2(M;\mathbb{R}) \cong \mathbb{R}$, so we should be able to get a harmc gradient. Using spherical coords, let $\varphi(r,\theta,\varphi) = \frac{-1}{r}$, which is constant on each cmpt of ∂M.



<u>HG:</u> Let $V = \nabla \varphi = \frac{1}{r^2}\frac{\partial}{\partial r}$

Since $\varphi = -\frac{1}{r}$ is harmc on M,

$$\nabla \cdot V = \nabla \cdot \nabla \varphi = \Delta \varphi = 0.$$