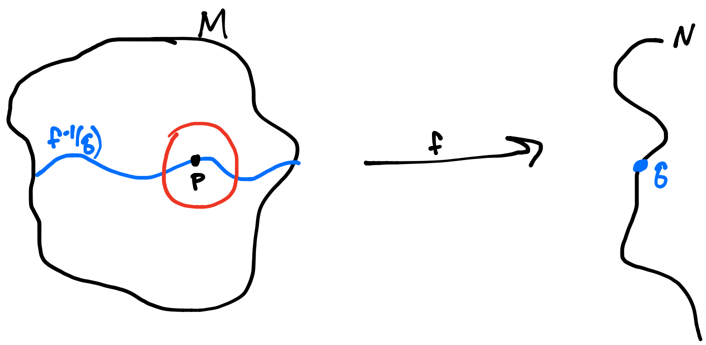
Math 8250: Day 4

Def: Let $f: M^m \to N^n$ be smooth. A point $p \in M$ is a **critical point** of f if $df_p: T_pM \to T_{f(p)}N$ is not surjective; then $f(p)$ is a **critical value** of f. A point $q \in N$ which is not a critical value is a **regular value**.

Thm: If $q \in N$ is regular then $f^{-1}(q)$ is a smooth submanifold of M of dimension $m-n$ (i.e., codimension n).



Ex:
- Define $f: \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \ldots, x_n) = \sum x_i^2$. Then all $r \neq 0$ are regular values, so the sphere $f^{-1}(r)$ of radius $\sqrt{r}$ in $\mathbb{R}^n$ is a smooth submfld of codimension 1 in $\mathbb{R}^n$.
- Define $f: GL(n,\mathbb{R}) \to (\text{symmetric } n\times n)$ by $f(A) = AA^T$. Then f is a submersion & $f^{-1}(I) = O(n)$.

Proof of Thm: Suppose (U, φ) a coord. chart at $p \in f^{-1}(q)$ and (V, ψ) a coord chart at q. Then $g = \psi^{-1} \circ f \circ \varphi : U \to \mathbb{R}^n$ and, by assumption, dg_0 is surjective, so after a linear change of coords can be written as the $n \times m$ matrix $[I_n \;\; 0]$. Then define $G(x_1, \ldots, x_m) = (g(x_1, \ldots, x_m), x_{n+1}, \ldots, x_m)$, which has differential $dG_0 = I_m$. By IVT G is a local diffeo. at 0, and $g \circ G^{-1}$ is the standard projection onto the first n coordinates.
Then in these coordinates $f^{-1}(q) \cap U = \varphi(0, \ldots, 0, x_{n+1}, \ldots, x_m)$, so $x_{n+1}, \ldots, x_m$ give local coords. for $f^{-1}(q)$. ∎

Vector Fields:

Def: A vector field X on a smooth mfd M is a smooth section of the tangent bundle; i.e., a C^∞ map $M \to TM$

Note: • If $\varphi: U \subset \mathbb{R}^n \to M$ is a coord. chart at $p \in M$, then

$$X(p) = \sum_{i}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

where each $a_i: U \to \mathbb{R}$ is a smooth function and $\{\frac{\partial}{\partial x_i}\}$ is the basis of T_pM associated w/ φ.

• As a map $C^\infty(M) \to C^\infty(M)$, a vector field acts as

$$(Xf)(p) = \sum_i a_i(p) \frac{\partial f}{\partial x_i}(p)$$

which notice really is a smooth function on M.

Def: If X & Y are smooth vector fields on M, then the Lie bracket of X & Y is a vector field $[X,Y]$ which is determined, as an operator on functions, by

$$[X,Y](f) := X(Yf) - Y(Xf).$$

In coordinates, if $X = \sum a_i \frac{\partial}{\partial x_i}$ & $Y = \sum b_i \frac{\partial}{\partial x_i}$, then

$$\begin{aligned}
[X,Y](f) &= X(Yf) - Y(Xf) \\
&= X\left(\sum b_i \frac{\partial f}{\partial x_i}\right) - Y\left(\sum a_i \frac{\partial f}{\partial x_i}\right) \\
&= \sum a_j\left(\frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + b_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right) - \sum b_j\left(\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right) \\
&= \sum\left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i} + \sum a_i b_j \cancel{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)} \\
&= \underbrace{\left(\sum\left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}\right)}_{\text{local coord. expression for } [X,Y]} f
\end{aligned}$$

Prop: If X, Y, Z are smooth vector fields on M and $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

① $[X,Y] = -[Y,X]$ (anticommutivity)

② $[aX+bY, Z] = a[X,Z] + b[Y,Z]$ (linearity)

③ $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity)

④ $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

Proof: Exercise. ∎