Math 8250: Day 39

Now that we've finished off the Frobenius Thm, I want to state (w/o proof) the version of the Hodge Theorem that holds for compact Riemannian mfds w/ bdry & then conclude some things about vector fields on compact domains in $\mathbb{R}^3$.

If M^n is a compact Riemannian mfd w/ $\partial M \neq \emptyset$, then of course the de Rham Theorem still holds:

$$H^p(M;\mathbb{R}) \cong H^p_{dR}(M)$$

Now, if we want the relative cohomology group $H^p(M,\partial M;\mathbb{R})$, we should consider only forms that vanish on the bdry: let

$$\Omega^p_D(M) = \{\omega \in \Omega^p(M) : i^*\omega = 0\}$$

where $i : \partial M \hookrightarrow M$ is the inclusion. Then we have the chain complex

$$\cdots \to \Omega^{p-1}_D(M) \xrightarrow{d} \Omega^p_D(M) \to \Omega^{p+1}_D(M) \to \cdots$$

since $i^* d\omega = d i^*\omega = 0$ for any $\omega \in \Omega^p_D(M)$.

Let $H^p_{dR,D}(M)$ be the homology of this complex.

Thm: $H^p(M,\partial M;\mathbb{R}) \cong H^p_{dR,D}(M)$.

Now, the Hodge Thm for closed mfds said that the orthogonal complement of the exact forms inside the space of closed forms gives the de Rham cohomology gp., since

$$\Omega^p(M) = \underbrace{d\Omega^{p-1}(M) \oplus \mathcal{H}^p(M)}_{\text{closed forms}} \oplus \delta\Omega^{p+1}(M)$$

(How did we know $\delta\Omega^{p+1}(M)$ was orthogonal to the closed forms? b/c if $\omega \in \Omega^p(M)$ closed & $\eta \in \Omega^{p+1}(M)$, then

$$\langle \omega, \delta\eta \rangle = \langle d\omega, \eta \rangle = \langle 0, \eta \rangle = 0.$$

Okay, but $\langle d\omega,\eta\rangle = \langle \omega, \delta\eta\rangle$ no longer holds on a mfld w/ bdry. Instead we have

Green's Formula: $\langle d\omega,\eta\rangle = \langle \omega, \delta\eta\rangle + \int_{\partial M} i^*\omega \wedge i^*\star\eta$

Proof: $\langle d\omega,\eta\rangle = \int_M d\omega\wedge\star\eta = \int_M d(\omega\wedge\star\eta) - (-1)^p\int_M \omega\wedge d\star\eta$

$$= \int_{\partial M} i^*(\omega\wedge\star\eta) - (-1)^p(-1)^{p(n-p)}\int_M \omega\wedge\star\star d\star\eta$$

$$= \int_{\partial M} i^*\omega\wedge i^*\star\eta - (-1)^{pn}\langle\omega,\star d\star\eta\rangle$$

$$= \int_{\partial M} i^*\omega\wedge i^*\star\eta - (-1)^{pn+n(p+1)+n+1}\langle\omega,\delta\eta\rangle$$

$$= \int_{\partial M} i^*\omega\wedge i^*\star\eta + \langle\omega,\delta\eta\rangle. \qquad \blacksquare$$

So the reasonable question is: what is the orth. complement of the exact forms inside the space of closed forms?

Well, if η is closed & orthogonal to the exact forms, then for any $\omega\in\Omega^{p-1}(M)$, we must have

$$0=\langle d\omega,\eta\rangle = \langle\omega,\delta\eta\rangle + \int_{\partial M} i^*\omega\wedge i^*\star\eta.$$

So we would at least expect that η must be co-closed & satisfy the bdry condition $i^*\star\eta=0$.

So η is closed, co-closed, & satisfies the Neumann boundary condition $i^*\star\eta=0$.

Define $\mathcal{H}^p_N(M) := \{\eta\in\Omega^p(M) : d\eta=0,\ \delta\eta=0,\ i^*\star\eta=0\}$.

Thm: $H^p(M;\mathbb{R}) \cong \mathcal{H}^p_N(M)$.

To get the relative cohomology, we want the orth. complement of the exact relative forms

$$\mathcal{E}^p_D(M) := \{\omega\in\Omega^p(M) : \omega = d\alpha,\ \alpha\in\Omega^{p-1}_D(M)\}$$

inside the space of closed p-forms satisfying the Dirichlet boundary condition $i^*\omega=0$.

If η is closed, satisfies the Dirichlet bdry condition, & is perpendicular to $\mathcal{E}^p_D(M)$, then $\forall\ d\alpha\in\mathcal{E}^p_D(M)$,

$$0=\langle d\alpha,\eta\rangle = \langle\alpha,\delta\eta\rangle + \int_{\partial M} \overset{0}{\cancel{i^*\alpha}}\wedge i^*\star\eta = \langle\alpha,\delta\eta\rangle,$$

So η should be co-closed in addition to being closed & satisfying the Dirichlet bdry condition. Then

Thm: If $\mathcal{H}^p_D(M) := \{\eta \in \Omega^p(M) : d\eta = 0,\ \delta\eta = 0,\ i^*\star\eta = 0\}$

then $H^p(M, \partial M; \mathbb{R}) \cong \mathcal{H}^p_D(M)$.

Then the result corresponding to the full Hodge thm is the following, proved by Morrey & Friedrichs in the 50's

Hodge-Morrey-Friedrichs Decomposition Thm: If M^n is a compact Riemannian mfld w $\partial M \neq \emptyset$, then

$$H^s\Omega^p(M) = H^s c\mathcal{E}^p_N(M) \oplus H^s\mathcal{H}^p(M) \oplus H^s\mathcal{E}^p_D(M)$$

for any $s \geq 0$ (e.g. $H^0 = L^2$ & H^1), where

$$c\mathcal{E}^p_N(M) := \{\omega \in \Omega^p(M) : \omega = \delta\beta \text{ for } \beta \in \Omega^{p+1}(M) \text{ s.t. } i^*\star\beta = 0\}$$

$$\mathcal{H}^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0,\ \delta\omega = 0\}$$

$$\mathcal{E}^p_D(M) := \{\omega \in \Omega^p(M) : \omega = d\alpha \text{ for } \alpha \in \Omega^{p-1}(M) \text{ s.t. } i^*\alpha = 0\}$$

The above is an orthogonal direct sum. Moreover, $H^s\mathcal{H}^p(M)$ has the further orthogonal decompositions

$$\begin{aligned} H^s\mathcal{H}^p(M) &= \mathcal{H}^p_N(M) \oplus H^s\mathcal{E}\mathcal{H}^p(M) \\ &= H^s c\mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}^p_D(M) \end{aligned}$$

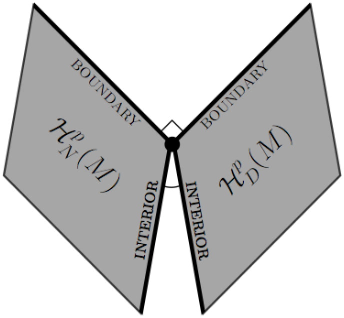
where

$$c\mathcal{E}\mathcal{H}^p(M) := \{\omega \in \mathcal{H}^p(M) : \omega = \delta\beta \text{ for } \beta \in \Omega^{p+1}(M)\}$$

$$\mathcal{E}\mathcal{H}^p(M) := \{\omega \in \mathcal{H}^p(M) : \omega = d\alpha \text{ for } \alpha \in \Omega^{p-1}(M)\}$$

It turns out that, although $\mathcal{H}^p_N(M) \cap \mathcal{H}^p_D(M) = \{0\}$, these two spaces are not generally orthogonal.

A more precise picture was given in the 2000s by DeTurck & Gluck, who showed that the picture is something like the following:



The boundary subspace of $\mathcal{H}^p_N(M)$ is the maximal subspace that is perpendicular to all of $\mathcal{H}^p_D(M)$;

Likewise, the boundary subspace of $\mathcal{H}^p_D(M)$ is the maximal subspace that is perpendicular to all of $\mathcal{H}^p_N(M)$.

Consequently, the angles b/w the interior subspaces are all acute (strictly b/w 0 & $\pi/2$).

DeTurck & Gluck called these Poincaré duality angles.

In examples they seem to be some sort of measure of how "close" a manifold is to being complete, but essentially no theorems are known (but if you have any ideas I would be very, very interested!).