Math 8250: Day 38

Frobenius Thm Version 2: A distribution $\mathcal{D}$ on M is involutive (& thus integrable) if & only if

$$d(\mathcal{I}(\mathcal{D})) \subset \mathcal{I}(\mathcal{D}).$$

Proof: Locally we can choose $\omega_1, \ldots, \omega_n$ that span $(T_x M)^*$ so that $\omega_{p+1}, \ldots, \omega_n$ generate $\mathcal{I}(\mathcal{D})$.

Let $X_1, \ldots, X_n$ be the dual vector fields; i.e.,

$$\omega_i(X_j) = \delta_{ij}.$$

Then $X_1, \ldots, X_p$ span $\mathcal{D}$ & so $\mathcal{D}$ is involutive iff $[X_i, X_j] \in \mathcal{D}$ $\forall$ $1 \leq i, j \leq p$.

But now, by Cartan's magic formula, for any v.f. X & any form ω,

$$L_X \omega = \iota_X d\omega + d\iota_X \omega \iff \iota_X d\omega = L_X \omega - d\iota_X \omega$$

Therefore, for $1 \leq i, j \leq p$ & $p+1 \leq k \leq n$,

$$\begin{aligned} d\omega_k(X_i, X_j) &= L_{X_i} \omega_k(X_j) - d(\omega_k(X_i))(X_j) \\ &= X_i(\cancelto{0}{\omega_k(X_j)}) - \omega_k([X_i, X_j]) - X_j(\cancelto{0}{\omega_k(X_i)}) \\ &= -\omega_k([X_i, X_j]) \end{aligned}$$

since $\omega \in \mathcal{I}(\mathcal{D})$ & $X_i, X_j \in \mathcal{D}$. But now $d\omega_k(X_i, X_j) = 0$ iff $\omega_k([X_i, X_j]) = 0$.

But of course the first is true for all k iff $\mathcal{I}(\mathcal{D})$ is a differential ideal & the second is true iff $\mathcal{D}$ is involutive. Therefore, these two are equivalent. $\blacksquare$

Note: We showed earlier in the semester that if $X, Y, Z \in \mathfrak{X}(S^3)$ are given by

$$X_p = ip, \quad Y_p = jp, \quad Z_p = kp \qquad \forall\ p \in S^3,$$

& α, β, γ are the dual 1-forms, then

$$\alpha \wedge d\alpha = 2\alpha \wedge \beta \wedge \gamma$$

So α is a contact 1-form. Then clearly the dual 2-plane distribution $\xi = \text{span}\{Y, Z\}$ is a non-integrable 2-plane distribution on S^3, called the standard contact structure.

Note: Since $\{\omega_i \wedge \omega_j\}$ spans $\Lambda^2 T_q M$ for each q, we can write

$$d\omega_k = \sum_{i<j} c_{ijk}\, \omega_i \wedge \omega_j = \sum_j \left(\sum_{i=1}^{j-1} c_{ijk}\, \omega_i \right) \wedge \omega_j = \sum_j \theta_{jk} \wedge \omega_j$$

for $\theta_{jk} \in \Omega^1(M)$. Now, if $k > p$ & $i_0, j_0 \le p$ are distinct, then

$$0 = d\omega_k(X_{i_0}, X_{j_0}) = \sum_j (\theta_{jk} \wedge \omega_j)(X_{i_0}, X_{j_0})$$
$$= \theta_{j_0 k}(X_{i_0}).$$

So we see that $d(\mathcal{I}(\mathcal{D})) \subset \mathcal{I}(\mathcal{D})$ iff

$$d\omega_k = \sum_{j>k} \theta_{jk} \wedge \omega_j .$$

Corollary: If $\omega_{p+1}, \dots, \omega_n$ are lin. ind. 1-forms in a nbhd of $p \in M$, then $\exists$ 1-forms θ_{jk} w/ $j, k > p$ s.t.

$$d\omega_k = \sum_j \theta_{jk} \wedge \omega_j$$

iff there are fcts f_{jk}, g_j s.t.

$$\omega_k = \sum_j f_{jk}\, dg_j .$$

Proof: Clearly, the first is true iff the distribution defined by the ideal $\mathcal{I} = \text{span}\{\omega_{p+1}, \dots, \omega_n\}$ is integrable.

If $x_1, \dots, x_n$ are local coords s.t. the coord. slices $\{x_{p+1}(q) = a_{p+1}, \dots, x_n(q) = a_n\}$ are integral mflds of $\mathcal{D}$, then $\mathcal{I} = \mathcal{I}(\mathcal{D}) = \text{span}\{dx_{p+1}, \dots, dx_n\}$, so the ω_k are linear combinations of the dx_j. Let $g_j = x_j$ be the coord. maps & we're done. $\blacksquare$