Math 8250: Day 37

Differential Ideals:

Suppose $\mathcal{D}$ is a d-dim'l dist. on M^n. Define $\mathcal{I}(\mathcal{D}) \subset \Omega(M)$ to be the ring of diff. forms that annihilate $\mathcal{D}$:

$$\mathcal{I}(\mathcal{D}) := \bigcup_p \{\omega \in \Omega^p(M) : \omega(X_1, \dots, X_p) = 0 \quad \forall X_1, \dots, X_p \in \mathcal{D}\}$$

Certainly if $\omega_1, \omega_2 \in \mathcal{I}(\mathcal{D})$, then $\omega_1 + \omega_2 \in \mathcal{I}(\mathcal{D})$ & $\omega_1 \wedge \omega_2 \in \mathcal{I}(\mathcal{D})$, so this really is a ring.

In fact, if $\omega \in \mathcal{I}(\mathcal{D}) \cap \Omega^p(M)$ & $\eta \in \Omega^q(M)$, then for any $X_1, \dots, X_{p+q} \in \mathcal{D}$,

$$(\omega \wedge \eta)(X_1, \dots, X_{p+q}) = \sum_{\sigma \in S_{(p,q)}} \operatorname{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(p)})\, \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})$$

$$= 0$$

so $\omega \wedge \eta \in \mathcal{I}(\mathcal{D})$. In other words, $\mathcal{I}(\mathcal{D})$ is an ideal.

We showed that any point $x \in M$ is contained in a coord. chart (U, φ) in which there are local coords $x_1, \dots, x_n$ s.t. $\mathcal{D} = \operatorname{span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\}$.

Let $\omega_1, \dots, \omega_n$ be the dual 1-forms. Then certainly $\omega_{d+1}, \dots, \omega_n$ are independent & generate $\mathcal{I}(\mathcal{D})$.

This proves the following lemma:

Lemma: For any d-dim'l dist $\mathcal{D}$ on M^n, the ideal $\mathcal{I}(\mathcal{D})$ is locally generated by $n-d$ ind. 1-forms $\omega_{d+1}, \dots, \omega_n$.

On the other hand...

Lemma: If $\mathcal{I} \subset \Omega(M)$ is an ideal locally generated by $n-d$ ind. 1-forms, then there is a d-dim'l dist. $\mathcal{D}$ on M s.t. $\mathcal{I} = \mathcal{I}(\mathcal{D})$.

Proof: Let $\omega_{d+1}, \dots, \omega_n$ be the independent 1-forms generating $\mathcal{I}$ in a nbhd of $x \in M$. Then define $\mathcal{D}(x) \subset T_x M$ to be the d-dim'l subspace annihilated by $\omega_{d+1}, \dots, \omega_n$. Then $\mathcal{D} = \bigcup_{x \in M} \mathcal{D}(x)$ is the desired distribution. ▨

Frobenius Thm Version 2: A distribution $\mathcal{D}$ on M is involutive (& thus integrable) if & only if

$$d(\mathcal{I}(\mathcal{D})) \subset \mathcal{I}(\mathcal{D}).$$

Def: An ideal $\mathcal{I} \subset \Omega(M)$ is called a **differential ideal** if $d(\mathcal{I}) \subset \mathcal{I}$.

Note that this version of the theorem makes it clear that a **contact structure** is very much not an integrable distribution: we can define a contact structure ξ on M^{2n+1} to be (locally) the kernel of a 1-form α s.t.

$$\alpha \wedge \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{n} \text{ is never zero.}$$

Clearly, $\mathcal{I}(\xi)$ is a $2n$-dim'l distribution on M, but the above condition implies that $d(\mathcal{I}(\xi)) \not\subset \mathcal{I}(\xi)$.

Why? Well let $X_1, \dots, X_{2n}$ be a (local) basis for ξ & complete to a basis $X_1, \dots, X_{2n+1}$ for the tangent space. Then

$$0 \neq (\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha)(X_1, \dots, X_{2n+1}) = \sum_i (-1)^{i-1} \alpha(X_i) (d\alpha \wedge \cdots \wedge d\alpha)(X_1, \dots, \hat{X}_i, \dots, X_{2n+1})$$

$$= \alpha(X_{2n+1}) (d\alpha \wedge \cdots \wedge d\alpha)(X_1, \dots, X_{2n})$$

since $\xi = \ker(\alpha)$ so $\alpha(X_i) = 0 \ \forall \ i \neq 2n+1$. But then we see that $d\alpha \wedge \cdots \wedge d\alpha \notin \mathcal{I}(\xi)$, though obviously $d\alpha \wedge \cdots \wedge d\alpha \in d(\mathcal{I}(\xi))$. Therefore, $d(\mathcal{I}(\xi)) \not\subset \mathcal{I}(\xi)$ & so ξ is not integrable.

(In fact, ξ is what's called **totally non-integrable**, meaning that no submanifolds of dimension $> n$ are tangent to ξ in a nbhd of any point).