Math 8250: Day 36

Here's an example of an involutive distribution where the vectors in the distribution don't commute:

Ex: Consider $M = S^3 - (N \cup S)$ where $N = \{(0,0,0,1)\}$ & $S = \{(0,0,0,-1)\}$ are the north & south poles.

Let $\mathcal{D} = \operatorname{span}\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\}$, which really is a distribution on $S^3 \subset \mathbb{R}^4$ since

$$\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\cdot\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right) = 0$$

& $$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\cdot\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right) = 0$$

Now,

$$\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right] = \cancel{yz\frac{\partial^2}{\partial x^2}} - \cancel{xy\frac{\partial^2}{\partial z\partial x}} - z\frac{\partial}{\partial y} - \cancel{xz\frac{\partial^2}{\partial x\partial y}} + \cancel{x^2\frac{\partial^2}{\partial z\partial y}}$$

$$-\left(\cancel{yz\frac{\partial^2}{\partial x^2}} - \cancel{xz\frac{\partial^2}{\partial y\partial x}} - y\frac{\partial}{\partial z} - \cancel{xy\frac{\partial^2}{\partial x\partial z}} + \cancel{x^2\frac{\partial^2}{\partial y\partial z}}\right)$$

$$= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$= \frac{-y}{x}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \frac{z}{x}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) \in \mathcal{D}(x,y,z,w),$$

so $\mathcal{D}$ is involutive. (What is the corresponding foliation?)

Frobenius Thm: Let $\mathcal{D}$ be a d-dim'l involutive distribution on M^n. Then for each $x \in M$ there exists an integral mfd of $\mathcal{D}$ passing through x.

Here's the intuition to have: Suppose $\mathcal{D}$ is spanned by vector fields $V_1, \dots, V_k$. Then thinking of the V_i as 1st-order differential operators on M, the collection $V_1, \dots, V_k$ determines a first order system of PDEs

$$V_1 u = 0, \ \dots, \ V_k u = 0.$$

Now, if $u: M \to \mathbb{R}$ is a solution of this system, then the level sets of u are tangent to $V_1, \dots, V_k$.

Then the question Frobenius was interested in was: when are there solutions $u_1, \dots, u_{n-k}$ s.t.

$\{\nabla u_1, \dots, \nabla u_{n-k}\}$ are linearly independent?

Notice that when there exist such $u_1, \ldots, u_{n-k}$, then the level sets of the map

$$(u_1, \ldots, u_{n-k}) : M \to \mathbb{R}^{n-k}$$

are integral manifolds of $\mathcal{D}$.

Proof of Frobenius Thm: I want to show that $\forall\, x \in M$ $\exists$ coordinate chart (U, φ) s.t.

① $\vec{0} \in U \subset \mathbb{R}^n$

② $\varphi(\vec{0}) = x$

③ For $\vec{a} = (a_1, \ldots, a_n) \in U$, the set

$$\{ p \in M : \varphi^{-1}(p)_{d+1} = a_{d+1}, \ldots, \varphi^{-1}(p)_n = a_n \}$$

is an integral mfd of $\mathcal{D}$.

Since everything is local, we may as well assume $M \subseteq \mathbb{R}^n$, $x = \vec{0}$, & $\mathcal{D}(x) = \mathcal{D}(\vec{0}) = \operatorname{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be the projection. Then $d\pi_0|_{\mathcal{D}} : \mathcal{D}(\vec{0}) \to T_{\vec{0}}\mathbb{R}^d$ is an isomorphism.

Therefore $d\pi$ is injective in a nbhd U of $\vec{0} \in \mathbb{R}^n$, so near $\vec{0}$ $\exists!$ $V_1, \ldots, V_d \in \mathcal{D}$ s.t.

$$d\pi_q V_i = \frac{\partial}{\partial x_i}\Big|_{\pi(q)} \quad \text{for all } q \text{ near } \vec{0} \text{ and all } i \in \{1, \ldots, d\}.$$

Now since $\mathcal{D}$ is involutive, $[V_i, V_j] \in \mathcal{D}$ $\forall$ i, j. On the other hand,

$$d\pi[V_i, V_j] = [d\pi V_i, d\pi V_j] = [\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}] = 0$$

Since $d\pi$ is an isomorphism, we see that $[V_i, V_j] = 0$. Therefore, we can change coords to get a new coord. system $\vec{y}$ s.t. $V_i = \frac{\partial}{\partial y_i}$ on U.

But then $\{\vec{y} \in U : y_{d+1} = a_{d+1}, \ldots, y_n = a_n\}$ (i.e. the level sets of map consisting of the last $n-d-1$ coords)

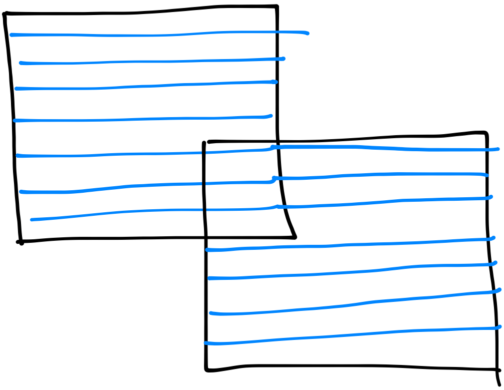
are integral mfds of $\mathcal{D} = \operatorname{span}\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_d}\}$. ∎

Notice that the above is really a local theorem. In fact, something more is true. If we call the subsets of U defined in the above proof **slices** of U, then it turns out that the slices in each coord. chart patch together nicely to give the following global result:

Global Frobenius Thm: Let $\mathcal{D}$ be a smooth d-dim'l distribution on M. Then M is **foliated** by a (disconnected) integral mfd for $\mathcal{D}$. The components of this foliation are called the **maximal integral manifolds** of $\mathcal{D}$.

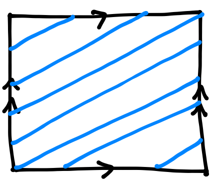
Def: A (disconnected) submfd $N^d \subset M^n$ is a **foliation** of M if each $x \in M$ is contained in (some component of) N and x is contained in a coord chart (U, φ) s.t. the components of $N \cap U$ are slices of $\varphi(U)$, meaning they're of the form $\{p \in \varphi(U) : \varphi^{-1}(p)_{d+1} = a_{d+1}, \dots, \varphi^{-1}(p)_n = a_n\}$.

Each component of the foliation N is called a **leaf** of the foliation.



Ex: For each choice of slope ρ, the torus $T^2 = S^1 \times S^1$ is foliated by lines of slope ρ.

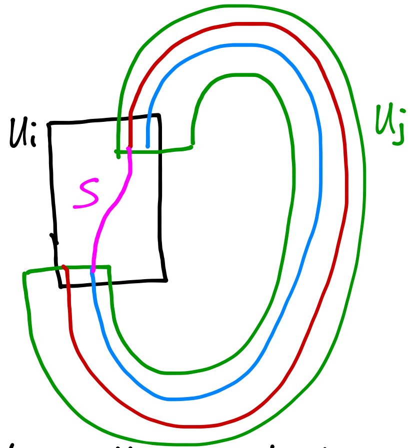
(When $\rho \notin \mathbb{Q}$, this is super-weird: every point in the quotient is **dense**!)



one leaf of the foliation of slope $\frac{3}{5}$.

- $\mathbb{R}^n - \{0\}$ is foliated by the concentric spheres.
- S^3 is foliated by the fibers of the Hopf fibration

Proof of Global Frobenius: Let $\{(U_i, \varphi_i)\}$ be a countable collection of coord. charts as in the local Frobenius thm. Then the worry is that the slices of one chart may not match up w/ the slices of another chart:



Suppose a slice S of U_i intersects more than one slice of U_j.

Since $S \cap U_j$ has at most countably many components & each component is contained in a single slice of U_j (by the local uniqueness of integral mflds), $S \cap U_j$ is contained in at most countably many slices of U_j.

Now, for any point $p \in M$, at most countably many slices from the countably many coord. charts of M are joined to p in this way. But then the union of these slices is a submfld of M. Moreover, for any other point $q \neq p$ the corresponding union is either equal to or totally disjoint from the union for p.

So then M is foliated by the disjoint union of all such submanifolds. ∎