Math 8250: Day 35

Distributions:

Speaking loosely for the moment, a distribution is a sub-bundle of the tangent bundle... at each point it consists of a subspace of the tangent space to that point.

Some common examples:
① The span of a vector field $V \in \mathfrak{X}(M)$.
② A plane field... for example, a contact structure.
③ The tangent spaces to a foliation of the manifold.

Def: A d-dimensional **distribution** $\mathcal{D}$ on a smooth mfld M^n is a (smooth) choice of subspace $\mathcal{D}(x) \subset T_xM$ for all $x \in M$.

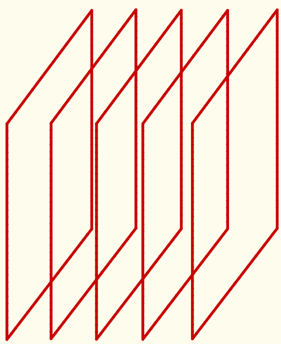
A vector field $V \in \mathfrak{X}(M)$ lies in $\mathcal{D}$ if $V_x \in \mathcal{D}(x)$ $\forall$ $x \in M$.

The distribution is **involutive** if $[V,W] \in \mathcal{D}$ for all smooth $V, W \in \mathcal{D}$.

Ex: ① On $\mathbb{R}^n$ let $\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}}\right\}$ for some multiindex $I = (i_1, \dots, i_k)$, $k < n$.

Then $\mathcal{D}$ is a k-dim'l distribution. Moreover, for all $1 \le p, q \le k$,

$$\left[\frac{\partial}{\partial x_{i_p}}, \frac{\partial}{\partial x_{i_q}}\right] = 0$$ since mixed partials commute on $\mathbb{R}^n$, so $\mathcal{D}$ is involutive.



The distribution $\text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\}$ on $\mathbb{R}^3$.

② Let $\mathcal{D}$ be the 1-dim'l distribution on $\mathbb{R}^2 - \{0\}$ which is perpendicular to the vector $\vec{r}$. Then at each point $(x,y) \in \mathbb{R}^2$, $\mathcal{D} = \text{span}\left\{V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right\}$. Obviously $[V,V] = 0$, so $\mathcal{D}$ is involutive.

③ Let $\mathcal{D}$ be the 2-dim'l distribution on $\mathbb{R}^3-\{0\}$ which is perpendicular to the vector $\vec{r}$. Then at each point $(x,y,z)\in\mathbb{R}^3$, $\mathcal{D}=\text{span}\{\underbrace{y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}}_{V},\ zx\underbrace{\frac{\partial}{\partial x}+zy\frac{\partial}{\partial y}-(x^2+y^2)}_{W}\frac{\partial}{\partial z}\}$

Then for any function $f:\mathbb{R}^3-\{0\}\to\mathbb{R}$,

$$[V,W](f)=V(W(f))-W(V(f))=\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\left(zx\frac{\partial f}{\partial x}+zy\frac{\partial f}{\partial y}-(x^2+y^2)\frac{\partial f}{\partial z}\right)$$

$$-\left(zx\frac{\partial}{\partial x}+zy\frac{\partial}{\partial y}-(x^2+y^2)\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}\right)$$

$$=\cancel{yz\frac{\partial f}{\partial x}}+\cancel{xyz\frac{\partial^2 f}{\partial x^2}}-\cancel{x^2z\frac{\partial^2 f}{\partial y\partial x}}+\cancel{y^2z\frac{\partial^2 f}{\partial x\partial y}}-\cancel{xz\frac{\partial f}{\partial y}}-\cancel{xyz\frac{\partial^2 f}{\partial y^2}}-\cancel{y(x^2+y^2)\frac{\partial^2 f}{\partial x\partial z}}-\cancel{2xy\frac{\partial f}{\partial z}}$$

$$+\cancel{x(x^2+y^2)\frac{\partial^2 f}{\partial y\partial z}}+\cancel{2xy\frac{\partial f}{\partial z}}-\cancel{xyz\frac{\partial^2 f}{\partial x^2}}+\cancel{xz\frac{\partial f}{\partial y}}+\cancel{x^2z\frac{\partial^2 f}{\partial x\partial y}}-\cancel{yz\frac{\partial f}{\partial x}}-\cancel{y^2z\frac{\partial^2 f}{\partial y\partial x}}$$

$$+\cancel{xyz\frac{\partial^2 f}{\partial y^2}}+\cancel{y(x^2+y^2)\frac{\partial^2 f}{\partial z\partial x}}-\cancel{x(x^2+y^2)\frac{\partial^2 f}{\partial z\partial y}}$$

$$=0,$$

so $[V,W]=0$ & we see that $\mathcal{D}$ is involutive.

④ Let $\xi=\text{span}\{\frac{\partial}{\partial y},\ \frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\}=\ker(dz-y\,dx)$ be a 2-dim'l distribution on $\mathbb{R}^3$.

(ξ is called the standard contact structure on $\mathbb{R}^3$) Then notice that for any $f:\mathbb{R}^3\to\mathbb{R}$,

$$\left[\frac{\partial}{\partial y},\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right](f)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right)(f)-\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right)\frac{\partial}{\partial y}(f)$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial z}\right)-\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right)\left(\frac{\partial f}{\partial y}\right)$$

$$=\frac{\partial^2 f}{\partial y\partial x}+y\frac{\partial^2 f}{\partial y\partial z}+\frac{\partial f}{\partial z}-\frac{\partial^2 f}{\partial x\partial y}-y\frac{\partial^2 f}{\partial z\partial y}$$

$$=\frac{\partial f}{\partial z}.$$

So $\left[\frac{\partial}{\partial y},\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right]=\frac{\partial}{\partial z}\notin\xi$, so ξ is <u>not</u> involutive (this is the defining characteristic of a contact structure on a 3-mfd: it is a non-involutive 2-dim'l distribution)

In the first 3 examples we could easily find integral mfds for the distributions (planes in ① & concentric circles/spheres in ②&③).

Def: The image of a smooth embedding $f: N \to M$ is an **integral manifold** of a distribution $\mathcal{D}$ on M if

$$df(T_pN) = \mathcal{D}(f(p)) \quad \text{for all } p \in N.$$

In other words, $df(TN) = Tf(N) = \mathcal{D}$.

Prop: If $\mathcal{D}$ is a smooth distribution on M s.t. at each $x \in M$ there is an integral mfd of $\mathcal{D}$ passing through x, then $\mathcal{D}$ is involutive.

Proof: If $V, W \in \mathcal{D}$, then at each point $x = f(p) \in f(N)$, $V_x = df_p V'$ & $W_x = df_p W'$.

Now, $[V', W']_p \in T_pN$ and $(df_p[V', W'])_x = [df_pV', df_pW']_x = [V, W]_x$

(this is a calculation). Therefore $\mathcal{D}$ is involutive. $\blacksquare$

The more surprising fact is that the converse is true, which is the content of the Frobenius theorem:

Frobenius Thm: Let $\mathcal{D}$ be a k-dim'l involutive distribution on M^n. Then for each $x \in M$ there exists an integral mfd of $\mathcal{D}$ passing thru x.

Here's the intuition to have: Suppose $\mathcal{D}$ is spanned by vector fields $V_1, \ldots, V_k$. Then thinking of the V_i as 1st-order differential operators on M, the collection $V_1, \ldots, V_k$ determines a first order system of PDEs

$$V_1 u = 0, \ \ldots, V_k u = 0.$$

Now, if $u: M \to \mathbb{R}$ is a solution of this system, then the level sets of u are tangent to $V_1, \ldots, V_k$.

Then the question Frobenius was interested in was: when are there solutions $u_1, \ldots, u_{n-k}$ s.t.

$\{\nabla u_1, \ldots, \nabla u_{n-k}\}$ are linearly independent?

Notice that when there exist such $u_1, \ldots, u_{n-k}$, then the level sets of the map

$$(u_1, \ldots, u_{n-k}): M \to \mathbb{R}^{n-k}$$

are integral manifolds of $\mathcal{D}$.

We can also think of integl mflds of distributions as (local) graphs of fctns.

e.g., Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth fctns. When does there exist $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}$ w/ $\vec{0} \in U$ s.t.

$$\varphi(\vec{0}) = z_0$$
$$\frac{\partial \varphi}{\partial x} = f$$
$$\frac{\partial \varphi}{\partial y} = g$$

Well, consider the distribution $\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x} + f\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + g\frac{\partial}{\partial z}\right\}$ on $\mathbb{R}^3$.

Then if φ is a sol'n of the above PDE, the graph of φ will be an intgl mfld of $\mathcal{D}$ passg thgh $(0,0,z_0)$.

So the Frobenius Thm says that such an intgl mfld exists $\Leftrightarrow$ $\left[\frac{\partial}{\partial x} + f\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + g\frac{\partial}{\partial z}\right] \in \mathcal{D}$

But now for any fctn $h : \mathbb{R}^3 \to \mathbb{R}$,

$$\left[\frac{\partial}{\partial x} + f\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + g\frac{\partial}{\partial z}\right]h = \cancel{\frac{\partial^2 h}{\partial x \partial y}} + \cancel{f\frac{\partial^2 h}{\partial z \partial y}} + \frac{\partial g}{\partial x}\frac{\partial h}{\partial z} + \cancel{g\frac{\partial^2 h}{\partial x \partial z}} + \cancel{fg\frac{\partial^2 h}{\partial z^2}}$$
$$- \cancel{\frac{\partial^2 h}{\partial y \partial x}} - \cancel{g\frac{\partial^2 h}{\partial z \partial x}} - \frac{\partial f}{\partial y}\frac{\partial h}{\partial z} - \cancel{f\frac{\partial^2 h}{\partial y \partial z}} - \cancel{fg\frac{\partial^2 h}{\partial z^2}}$$
$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\frac{\partial h}{\partial z}$$

so $\left[\frac{\partial}{\partial x} + f\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + g\frac{\partial}{\partial z}\right] = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\frac{\partial}{\partial z}$, which is in $\mathcal{D} \Leftrightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.

Of course, we already knew this was a necessary condition since, if φ is a sol'n, it must be the case that

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \varphi}{\partial x} = \frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial y} = \frac{\partial g}{\partial x},$$

but the sufficiency of the condition is the real content of the Frobenius Thm.