Math 8250: Day 34

Thm: If $\eta \in \mathcal{H}^p(M)^\perp$ then there exists a unique $\omega \in \left(H^1\mathcal{H}^p(M)\right)^\perp \subset H^1\Omega^p(M)$ s.t.

$$\langle d\omega, d\xi\rangle + \langle \delta\omega, \delta\xi\rangle = \langle \eta, \xi\rangle \quad \text{for all } \xi \in H^1\Omega^p(M). \qquad (*)$$

In particular, ω is a weak soln to $\Delta\omega = \eta$ since $\langle \Delta\omega, \xi\rangle = \langle d\omega, d\xi\rangle + \langle \delta\omega, \delta\xi\rangle = \langle \eta, \xi\rangle \;\forall \xi \in H^1\Omega^p(M)$.

The key tool in the proof is the Dirichlet Integral

$$\mathcal{D}(\omega,\xi) := \langle d\omega, d\xi\rangle + \langle \delta\omega, \delta\xi\rangle = \langle\langle \omega,\xi\rangle\rangle - \langle \omega,\xi\rangle.$$

Then if we can show that the bilinear form is • bounded (i.e. $\exists\, C>0$ s.t. $\mathcal{D}(\omega,\eta) \leq C\|\omega\|_1\|\eta\|_1$)

and • coercive ($\exists\, c>0$ s.t. $\mathcal{D}(\omega,\omega) \geq c\|\omega\|_1^2$)

on the space $\left(H^1\mathcal{H}^p(M)\right)^\perp$, then we can use the

Lax-Milgram Lemma: If B is a bilinear form on a Hilbert space $\mathbb{H}$ w/ inner product $\langle\,,\rangle_{\mathbb{H}}$ & induced norm $\|\cdot\|_{\mathbb{H}}$ that is bounded & coercive, then for any bounded linear functional $F \in \mathbb{H}^*$ $\exists!\, u \in \mathbb{H}$ s.t.

$$F(v) = \langle u, v\rangle_{\mathbb{H}} \qquad \forall\, v \in \mathbb{H}.$$

Proof: HW. ∎

Lemma: $\mathcal{D}$ is bdd & coercive on $\left(H^1\mathcal{H}^p(M)\right)^\perp$.

Proof: Boundedness is obvious: For $\omega,\eta \in \left(H^1\mathcal{H}^p(M)\right)^\perp$,

$$\mathcal{D}(\omega,\eta) = \langle d\omega, d\eta\rangle + \langle \delta\omega, \delta\eta\rangle \leq \|d\omega\|_0\|d\eta\|_0 + \|\delta\omega\|_0\|\delta\eta\|_0 \leq C_1\|\omega\|_0\|\eta\|_0 + C_2\|\omega\|_0\|\eta\|_0$$
$$= (C_1+C_2)\|\omega\|_0\|\eta\|_0$$

by the continuity of d & δ.

OTOH, for coerciveness, let $\{\omega_k\}$ be a minimizing sequence for $\mathcal{D}$ among elts of $\left(H^1\mathcal{H}^p(M)\right)^\perp$ w/ unit L^2-norm.

In other words, $\|\omega_k\|_0 = 1 \ \forall k$ & $\mathcal{D}(\omega_k,\omega_k) \to \inf\limits_{\substack{\omega\in (H^1\mathcal{H}^p(M))^\perp \\ \text{s.t. } \|\omega\|_0 = 1}} \mathcal{D}(\omega,\omega)$.

Then $\|\omega_k\|_1 \le C$ for some $C>1$, so $\{\omega_k\}$ is a bdd seq. in $(H^1\mathcal{H}^p(M))^\perp$. Therefore, by the Rellich Lemma, a subsequence

$\{\omega_{k_j}\}$ converges in L^2 to some $\omega \in (\mathcal{H}^p(M))^\perp$. But then an easy functional analysis argument implies $\omega \in (H^1\mathcal{H}^p(M))^\perp$:

Boundedness implies $\omega_{k_j} \rightharpoonup \hat{\omega} \in (H^1\mathcal{H}^p(M))^\perp \subset (\mathcal{H}^p(M))^\perp$. But then for any $F \in ((\mathcal{H}^p(M))^\perp)^*$,

$$|F(\hat{\omega}-\omega)| \le \left(|F(\hat{\omega}-\omega_{k_j})| + |F(\omega_{k_j}-\omega)|\right) \to 0, \text{ so } \hat{\omega} = \omega.$$

Now, since $\|\omega_{k_j}\|_0 = 1 \ \forall k_j$, we know $\|\omega\|_0 = 1$. Moreover, since $\omega \in (\mathcal{H}^p(M))^\perp$,

$$0 < \|d\omega\|_0^2 + \|\delta\omega\|_0^2 = \mathcal{D}(\omega,\omega).$$

Therefore, $\inf\limits_{\substack{\eta\in(H^1\mathcal{H}^p(M))^\perp \\ \|\eta\|=1}} \mathcal{D}(\eta,\eta) = \mathcal{D}(\omega,\omega) > 0$, so for all $\eta \in (H^1\mathcal{H}^p(M))^\perp$ s.t. $\|\eta\|_0 = 1$, $\mathcal{D}(\eta,\eta) \ge \mathcal{D}(\omega,\omega)$.

But then

$$\mathcal{D}(\eta,\eta) = \|\eta\|_0^2\, \mathcal{D}\left(\frac{\eta}{\|\eta\|_0}, \frac{\eta}{\|\eta\|_0}\right) \ge \|\eta\|_0^2\, \mathcal{D}(\omega,\omega) \qquad \text{for all } \eta \in (H^1\mathcal{H}^p(M))^\perp$$

or, equivalently,
$$\|\eta\|_0^2 \le \frac{1}{\mathcal{D}(\omega,\omega)}\mathcal{D}(\eta,\eta).$$

But then

$$\|\eta\|_1^2 = \langle \eta,\eta\rangle + \langle d\eta, d\eta\rangle + \langle \delta\eta,\delta\eta\rangle = \|\eta\|_0^2 + \mathcal{D}(\eta,\eta) \le \left(\frac{1}{\mathcal{D}(\omega,\omega)} + 1\right)\mathcal{D}(\eta,\eta)$$

for all $\eta \in (H^1\mathcal{H}^p(M))^\perp$, so $\mathcal{D}$ is indeed coercive on $(H^1\mathcal{H}^p(M))^\perp$ ∎

Proof of Thm: For $\eta \in \mathcal{H}^p(M)^\perp \subset L^2\Omega^p(M)$ define $F : (H^1\mathcal{H}^p(M))^\perp \to \mathbb{R}$ by

$$F(\tilde{\xi}) = \langle \eta, \tilde{\xi} \rangle$$

Then F is certainly a bounded linear functional since $\langle \eta,\tilde{\xi}\rangle \le \|\eta\|_0\|\tilde{\xi}\|_0 \le \|\eta\|_1\|\tilde{\xi}\|_1$ by definition.

Since the Dirichlet integral $\mathcal{D}$ is bdd & coercive on the Hilbert space $(H^1\mathcal{H}^p(M))^\perp$, the Lax-Milgram Lemma implies that there exists $\omega \in (H^1\mathcal{H}^p(M))^\perp$ s.t.

$$\langle \eta, \tilde{\xi} \rangle = F(\tilde{\xi}) = D(\omega, \tilde{\xi}) = \langle d\omega, d\tilde{\xi} \rangle + \langle \delta\omega, \delta\tilde{\xi} \rangle \qquad \forall\, \tilde{\xi} \in (H^1\mathcal{H}^p(M))^\perp$$

So it remains only to extend to all of $H^1\Omega^p(M)$. For $\xi \in H^1\Omega^p(M) \subset L^2\Omega^p(M)$, we can write

$$\xi = \lambda + \tilde{\xi} \in \mathcal{H}^p(M) \oplus \mathcal{H}^p(M)^\perp.$$

Then $\langle \eta, \xi \rangle = \langle \eta, \lambda + \tilde{\xi} \rangle = \langle \eta, \tilde{\xi} \rangle$ and

$$D(\eta, \xi) = \langle d\eta, d(\lambda + \tilde{\xi}) \rangle + \langle \delta\eta, \delta(\lambda + \tilde{\xi}) \rangle = \langle d\eta, d\tilde{\xi} \rangle + \langle \delta\eta, \delta\tilde{\xi} \rangle = D(\eta, \tilde{\xi}),$$

so indeed

$$\langle \eta, \xi \rangle = \langle \eta, \tilde{\xi} \rangle = D(\eta, \tilde{\xi}) = D(\eta, \xi) \qquad \forall\, \xi \in H^1\Omega^p(M).$$

To prove uniqueness, suppose ω' is another solution of ⊛. Then

$$\langle d(\omega - \omega'), d\xi \rangle + \langle \delta(\omega - \omega'), \delta\xi \rangle = 0 \qquad \forall\, \xi \in H^1\Omega^p(M).$$

In particular, if $\xi = \omega - \omega'$ then we see that $D(\omega - \omega', \omega - \omega') = 0$, so $\omega - \omega' \in \mathcal{H}^p(M)$. But $\omega, \omega' \in (H^1\mathcal{H}^p(M))^\perp$, so $\omega - \omega' = 0$.

On the other hand, if η & η' have the same solution ω, then

$$\langle \eta - \eta', \xi \rangle = 0 \qquad \forall\, \xi \in H^1\Omega^p(M).$$

Since $H^1\Omega^p(M) \subset L^2\Omega^p(M)$ is dense, this implies $\eta - \eta' = 0$.

This then completes the proof of the theorem and, with it, the Hodge theorem. ▨