Math 8250: Day 33

Sobolev Spaces of forms:

Let M^n be a closed oriented mfld. On $\Omega^p(M)$ we introduce the new scalar product:

$$\langle\langle \omega,\eta\rangle\rangle := \langle \omega,\eta\rangle + \langle d\omega, d\eta\rangle + \langle \delta\omega,\delta\eta\rangle$$

and define the 1-norm

$$\|\omega\|_1 = \sqrt{\langle\langle\omega,\omega\rangle\rangle}.$$

Define the Sobolev space $W^{1,2}\Omega^p(M) = H^1\Omega^p(M)$ to be the completion of $\Omega^p(M)$ w.r.t. this norm

(We let $L^2\Omega^p(M)$ be the completion w.r.t. the usual norm)

Now, for every $x\in M$ $\exists$ an open nbhd U of x & a diffeo.

$$\varphi: \Lambda^p(M)|_U \to V\times\mathbb{R}^d$$

where $V\subset\mathbb{R}^n$ is open & $d=\binom{n}{p}$ is the dimension of the fibers of $\Lambda^p(M)=\bigcup_{y\in M}\Lambda^p((T_yM)^*)$

& the map is a bundle map.

Now, for $\omega\in\Omega^p(M)$, we can interpret $\omega|_U$ as a section of $\Lambda^p(M)|_U$. Then $\varphi(\omega)$ is a section of the (trivial) bundle $V\times\mathbb{R}^d$, which is to say, a smooth map $V\to\mathbb{R}^d$.

We can extend our definition of the Sobolev 1-norm for functions to such maps; in particular, if $\vec F: V\to\mathbb{R}^d$, then

$$\|\vec F\|^2_{H^1_{loc}(V)} = \sum_{|\alpha|\le1}\|D^\alpha \vec F\|^2_{L^2} = \int_V \vec F\cdot\vec F + \int_V \frac{\partial\vec F}{\partial x_i}\cdot\frac{\partial \vec F}{\partial x_i}$$

Lemma: On any $U'\Subset U$ (i.e. any relatively compact subset of U) the norms

$$\|\omega\|_1 \quad \& \quad \|\varphi(\omega)\|_{H^1_{loc}(V')}$$

are equivalent, where $\pi: V\times\mathbb{R}^d\to V$ is the projection & $V'=\pi(\varphi(U'))$.

Proof: Computation.

Therefore, all the results for Euclidean Sobolev spaces carry over to our norm $\|\cdot\|_1$ on the Riemannian mfd M.

In particular, Rellich's Lemma holds in this setting & says:

Rellich Lemma: $H^1\Omega^p(M) \longrightarrow L^2\Omega^p(M)$ is a compact embedding. In particular, if $\{\omega_n\}$ is a bounded sequence in $H^1\Omega^p(M)$ (i.e. $\|\omega_n\|_1 \leq K$ for all n) then some subsequence converges in $L^2\Omega^p(M)$.

Prop: $H^1\mathcal{H}^p(M) := \{\omega \in H^1\Omega^p(M) : d\omega = 0, \delta\omega = 0\}$ is finite-dimensional.

Proof: Notice, first of all, that for $\omega \in H^1\mathcal{H}^p(M)$,

$$\|\omega\|_1^2 = \|\omega\|_0^2 + \|d\omega\|_0^2 + \|\delta\omega\|_0^2 = \|\omega\|_0^2,$$

so the 1-norm & the 0-norm (i.e. L^2 norm) agree on $H^1\mathcal{H}^p(M)$. But then that means the unit disk in $H^1\mathcal{H}^p(M)$, which is the set

$$D_1 := \{\omega \in H^1\mathcal{H}^p(M) : \|\omega\|_1 \leq 1\}$$

is closed in L^2. In fact, by the Rellich Lemma every sequence in D_1 has a subseq. which converges in L^2 & hence in H^1, so D_1 is compact.

But now it's a standard fact in functional analysis that the unit ball in a Banach space is compact $\Leftrightarrow$ the Banach space is finite-dim'l, so we conclude that $H^1\mathcal{H}^p(M)$ is finite-dim'l. $\blacksquare$

Cor: $(H^1\mathcal{H}^p(M))^\perp \subset H^1\Omega^p(M)$ is complete (and hence a Hilbert space).

Prop: The s-norm closure of the (finite-dim'l) kernel of an elliptic operator is independent of s.

Cor: $L^2\mathcal{H}^p(M) = H^1\mathcal{H}^p(M) = H^\infty\mathcal{H}^p(M) = \mathcal{H}^p(M)$.

Cor: $L^2\Omega^p(M) = \mathcal{H}^p(M) \oplus \mathcal{H}^p(M)^\perp$.

Now, all we need to complete the proof of the Hodge Theorem is that $\Delta\omega = \eta$ is weakly solvable if $\eta \in \mathcal{H}^p(M)^\perp \subset L^2\Omega^p(M)$ since we can then invoke elliptic regularity to conclude that any weak solution is a strong solution.

Thm: If $\eta \in \mathcal{H}^p(M)^\perp$ then there exists a unique $\omega \in (H^1\mathcal{H}^p(M))^\perp \subset H^1\Omega^p(M)$ s.t.

$$\langle d\omega, d\xi\rangle + \langle \delta\omega, \delta\xi\rangle = \langle \eta, \xi\rangle \quad \text{for all } \xi \in H^1\Omega^p(M). \qquad (*)$$

In particular, ω is a weak soln to $\Delta\omega = \eta$ since $\langle \Delta\omega, \xi\rangle = \langle d\omega, d\xi\rangle + \langle \delta\omega, \delta\xi\rangle = \langle \eta, \xi\rangle \ \forall \xi \in H^1\Omega^p(M)$.

The key tool in the proof is the Dirichlet Integral

$$\mathcal{D}(\omega, \xi) := \langle d\omega, d\xi\rangle + \langle \delta\omega, \delta\xi\rangle = \langle\langle \omega, \xi\rangle\rangle - \langle \omega, \xi\rangle.$$

Then if we can show that the bilinear form is • bounded (i.e. $\exists\, C > 0$ s.t. $\mathcal{D}(\omega, \eta) \leq C\|\omega\|_1 \|\eta\|_1$)

and • coercive ($\exists\, c > 0$ s.t. $\mathcal{D}(\omega, \omega) \geq c\|\omega\|_1^2$)

on the space $(H^1\mathcal{H}^p(M))^\perp$, then we can use the

Lax-Milgram Lemma: If B is a bilinear form on a Hilbert space $\mathbb{H}$ w/ inner product $\langle\, ,\rangle_{\mathbb{H}}$ & induced norm $\|\cdot\|_{\mathbb{H}}$ that is bounded & coercive, then for any bounded linear functional $\mathcal{F} \in \mathbb{H}^*$ $\exists!$ $u \in \mathbb{H}$ s.t.

$$\mathcal{F}(v) = \langle u, v\rangle_{\mathbb{H}} \quad \forall\, v \in \mathbb{H}.$$