Math 8250: Day 32

Ellipticity

First, define a differential operator L on $U \subset \mathbb{R}^n$ to be a map $L: C^\infty(U) \to C^\infty(U)$ given by

$$(*) \qquad Lu = \sum_{|\alpha| \le r} a_\alpha D^\alpha u \qquad \text{w/ } a_\alpha \in C^\infty(U).$$

The number r is called the order of L.

Ex: $\cdot\ \Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is an order 2 differential operator.

We can extend the definition by letting L be an $m \times n$ matrix whose entries are of the form $(*)$; then L is a map $C^\infty(U \subset \mathbb{R}^n; \mathbb{R}^m) \to C^\infty(U \subset \mathbb{R}^n; \mathbb{R}^m)$ & likewise for complex-valued functions.

Ex: Under this extended defn, ∇, $\nabla \times$, & $\nabla \cdot$ are order 1 differential operators & the vector Laplacian is an order 2 diff. operator.

Given a diff. operator L of order r, define the symbol of L to be the map $\sigma_L: \underbrace{U \times \mathbb{R}^n}_{= T^*U} \to \mathbb{R}$ given by

$$\sigma_L(x, \xi) := \sum_{|\alpha| = r} a_\alpha(x) \xi^\alpha.$$

Ex: The symbol of Δ on $\mathbb{R}^n$ is just

$$\sigma_\Delta(x, \xi) = -\sum_{i=1}^n \xi_i^2 = -|\xi|^2.$$

In the more general case where L is a matrix, the principal symbol σ_L is also a matrix.

Def: A differential operator L is elliptic if $\sigma_L(x, \xi) \neq 0$ for all $x \in U$ & $\xi \in \mathbb{R}^n - \{0\}$.

(In higher dimensions when σ_L is a matrix, the condition is that the matrix is nonsingular.)

The operator L is called uniformly elliptic if $\exists\ 0 < c \le C$ s.t.

$$c|\xi|^2 \le \sigma_L(x, \xi) \le C|\xi|^2$$

for all $x \in U, \xi \in \mathbb{R}^n$.

Notice, in particular, that the Laplacian is trivially uniformly elliptic.

Fundamental Inequality: If L is an elliptic operator on $U \subseteq \mathbb{R}^n$ of order r, then for each s $\exists$ $c>0$ s.t.

$$\|u\|_{s+r} \leq c\left(\|Lu\|_s + \|u\|_s\right)$$

for all $u \in H_c^{s+r}(U)$.

Proof when L has constant coefficients and consists of the leading term only:

Since C^∞ functions are dense in $H_c^{s+r}(U)$, it suffices to prove it for smooth u.

First, note that by rescaling the compact support of u & extending u by zero we can make u a periodic function, so u has a Fourier Series $\quad u = \sum u_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$.

Now, the goal is to show that $\|Lu\|_s^2 \geq c \sum_{\vec{k}} |\vec{k}|^{2r}(1+|\vec{k}|^2)^s |u_{\vec{k}}|^2$ for some $c>0$. (**)

If that's true, then we have that

$$\begin{aligned}(\|Lu\|_s + \|u\|_s)^2 \geq \|Lu\|_s^2 + \|u\|_s^2 &\geq c\sum_{\vec{k}} |\vec{k}|^{2r}(1+|\vec{k}|^2)^s|u_{\vec{k}}|^2 + \sum(1+|\vec{k}|^2)^s|u_{\vec{k}}|^2 \\ &= \sum \left(c|\vec{k}|^{2r}+1\right)\left(1+|\vec{k}|^2\right)^s|u_{\vec{k}}|^2 \\ &\geq c' \sum (1+|\vec{k}|^2)^{s+r}|u_{\vec{k}}|^2 \\ &= c'\|u\|_{s+r}^2\end{aligned}$$

and the theorem follows by dividing both sides by c'.

Okay, but now proving (**) is easy. First, notice that, since L has no lower-order terms & has const. coeffs

$$\|Lu\|_s^2 = \sum_{\vec{k}} (1+|\vec{k}|^2)^s \left|\sigma_L(t,\vec{k})\, u_{\vec{k}}(t)\right|^2.$$

Now, if $\vec{k}$ & $u_{\vec{k}}(t)$ are nonzero, $|\sigma_L(t,\vec{k})u_{\vec{k}}(t)|^2 > 0$ since L is elliptic. If we rescale for the moment so that $|\vec{k}| = 1 = |u_{\vec{k}}(t)|$, then by compactness of the unit sphere we know that

$$|\sigma_L(t,\vec{n})u_{\vec{n}}(t)|^2 \geq c$$

for some $c>0$. In general, of course, we have to pay by a factor of $|\vec{n}|^{2r}|u_{\vec{n}}(t)|^2$ to rescale, so

$$|\sigma_L(t,\vec{n})u_{\vec{n}}(t)|^2 \geq c|\vec{n}|^{2r}|u_{\vec{n}}(t)|^2.$$

But then we have

$$\|Lu\|_s^2 = \sum_{\vec{n}} (1+|\vec{n}|^2)^s |\sigma_L(t,\vec{n})u_{\vec{n}}(t)|^2 \geq c\sum |\vec{n}|^{2r}(1+|\vec{n}|^2)^s|u_{\vec{n}}|^2$$

as desired. ∎

In particular, notice that we now have

$$\|u\|_{s+r} \leq c\|(I+L)u\|_s$$

for each elliptic L of order r. Therefore, when $I+L$ is self-adjoint & invertible (e.g. when L is positive) the above implies that $T=(I+\Delta)^{-1}$ is a compact self-adjoint operator & hence we can apply the spectral theorem to get a decomposition

$$L^2(U) = (\ker L) \oplus (E(\lambda_i))$$

where the λ_i are the eigenvalues of L. This yields the (self-adjoint case of the)

Fredholm Theorem for Elliptic Operators: If M is a cpt mfd &

$$L: C^\infty(M) \to C^\infty(M)$$

is an elliptic diff'l operator, the kernel of L is finite-dim'l & the PDE

$$Lu = \varphi$$

is solvable $\iff$ $\langle \varphi, h\rangle = 0$ for all $h \in \ker L^*$.

Finally & most importantly, we have

Elliptic Regularity: Suppose that $U \subset \mathbb{R}^n$ & $\varphi \in H^s_c(U)$. If $u \in L^2_c(U)$ is a weak soln of the eqn

$$\Delta u = \varphi$$

(meaning that $\langle u, \Delta\psi \rangle = \langle \varphi, \psi \rangle$ for all $\psi \in C^\infty(U)$), then $u \in H^{s+2}_c(U)$.