## Math 8250: Day 30

Recall that last time we proved (most of) the Sobolev Lemma, which says that

$$W^{s+r,p}(T^d) \subset C^s(T^d)$$

for all $s \geq 0, r > 0$ & $p > \frac{d}{r}$. In particular, $H^{s+r}(T^d) \subset C^s(T^d)$ for $r > \frac{d}{2}$.

As a consequence, we also have that $C^\infty(T^d) \subset H^\infty(T^d)$. In fact, $C^\infty(T^d) = H^\infty(T^d)$.

Also, while proving the Sobolev Lemma we saw that $\sum |c_n| < C\|u\|_r$ for $r > \frac{d}{2}$. Hence,

$$|u(\vec{t})| = \left|\sum c_n e^{i\vec{n}\cdot\vec{t}}\right| \leq \sum |c_n| \leq C\|u\|_r$$

for all $\vec{t} \in T^d$, so $\sup\limits_{\vec{t}\in T^d} |u(\vec{t})| \leq C\|u\|_r$ for all $r > d/2$. Now, from the definition of the $s$-norm,

$$\|D^\alpha u\|_s \leq C_\alpha \|u\|_{s+|\alpha|}.$$

Therefore, for any $\alpha = (\alpha_1, \dots, \alpha_d)$ & $u \in C^\infty(T^d)$,

(*) $$\sup_{\vec{t}\in T^d} |D^\alpha u(\vec{t})| \leq C\|D^\alpha u\|_r \leq C'_\alpha \|u\|_{r+|\alpha|} \quad \text{for } r > d/2.$$

**Rellich Lemma:** For $s > r$, the embedding $H^s(T^d) \subset H^r(T^d)$ is compact.

Recall that a map $X \to Y$ b/w Banach spaces is compact if the image of any bdd subset of $X$ is relatively compact in $Y$. Equivalently, the image of every bounded sequence in $X$ has a subsequence which converges in $Y$.

So the Rellich Lemma says that if $\{u_k\}$ is a sequence s.t. $\|u_k\|_s < C$ for some $C$, then some subsequence converges to $u \in H^r(T^d)$.

For our purposes, it will be most useful to know that if $\|u_k\|_1 < C$, then (a subsequence of) $\{u_k\}$ converges to $u \in L^2$.

**Proof of Rellich Lemma:** Let $\{u_k\}$ be a bdd seq. in $H^s(T^d)$. For all $k$ we have

$$\sum_{\vec{n}\in\mathbb{Z}^d} (1+|\vec{n}|^2)^r |c_{\vec{n},k}|^2 \leq \sum_{\vec{n}\in\mathbb{Z}^d} (1+|\vec{n}|^2)^s |c_{\vec{n},k}|^2 < C \qquad (**)$$

So for each lattice point $\vec{n}$ we know that $\{(1+|\vec{n}|^2)^{r/2} c_{\vec{n},k}\}$ is bdd & thus has a Cauchy subsequence.

Then diagonalizing yields a subsequence of $\{u_k\}$ (which we just call $\{u_k\}$) so that $\{(1+|\vec{n}|^2)^{r/2} c_{\vec{n},k}\}$ is Cauchy for every $\vec{n}$.

Now the goal is to see that this subsequence $\{u_k\}$ is itself Cauchy, so we analyze

$$\|u_k - u_l\|_r^2 = \sum_{\vec{n}} (1+|\vec{n}|^2)^r |c_{\vec{n},k} - c_{\vec{n},l}|^2$$

Now, when $|\vec{n}|$ is smaller than some fixed number, we get a finite sum, which we can make arbitrarily small by choosing $k$ & $l$ sufficiently big.

On the other hand, when $|\vec{n}|$ is big we can make $\frac{1}{(1+|\vec{n}|^2)^{s-r}}$ arbitrarily small, factor it out, & use the second inequality in $\star\star$.

So here we go: let $\varepsilon > 0$ & choose $R$ large enough that

$$\frac{2C}{(1+|\vec{n}|^2)^{s-r}} < \varepsilon/2 \quad \text{for } |\vec{n}| > R.$$

Also, choose $m \in \mathbb{N}$ s.t.

$$\sum_{|\vec{n}| \le R} (1+|\vec{n}|^2)^r |c_{\vec{n},k} - c_{\vec{n},l}| < \varepsilon/2 \qquad \text{for } k,l \ge m$$

(which again we can do because of the choice of the subsequence $\{u_k\}$ & because the above sum is finite).

Then for $k,l \ge m$,

$$\|u_k - u_l\|_r^2 = \sum_{\vec{n}} (1+|\vec{n}|^2)^r |c_{\vec{n},k} - c_{\vec{n},l}|^2 = \sum_{|\vec{n}| \le R} (1+|\vec{n}|^2)^r |c_{\vec{n},k} - c_{\vec{n},l}|^2 + \sum_{|\vec{n}| > R} \frac{(1+|\vec{n}|^2)^s}{(1+|\vec{n}|^2)^{s-r}} |c_{\vec{n},k} - c_{\vec{n},l}|^2$$

$$< \frac{\varepsilon}{2} + \sum_{|\vec{n}| > R} \frac{\varepsilon}{4C} \cdot (1+|\vec{n}|^2)^s |c_{\vec{n},k} - c_{\vec{n},l}|^2$$

$$< \varepsilon/2 + \frac{\varepsilon}{4C} \cdot 2C = \varepsilon.$$

So indeed $\{u_k\}$ is Cauchy. $\blacksquare$

## Digression on distributions

A *distribution* on $\mathbb{T}^d$ is a linear functional

$$\lambda : C^\infty(\mathbb{T}^d) \to \mathbb{C}$$

which is continuous in the sense that

$$|\lambda(u)| \le C_\lambda \sup_{\substack{|\alpha|\le k \\ \vec{t}\in\mathbb{T}^d}} |D^\alpha u(\vec{t})|$$

for some $k$. Then from our estimate ⊛ which said $\sup_{\vec{t}\in\mathbb{T}^d} |D^\alpha u(\vec{t})| \le C'_\alpha \|u\|_{r+|\alpha|}$

for $r > d/2$, we see that each $\lambda$ is a cont. linear function on $H^s(\mathbb{T}^d)$ for some $s$ (e.g. $s = \lfloor d/2 \rfloor + 1 + k$)

Now we can represent each distribution by a Fourier series

$$\sum \lambda_{\vec{n}} e^{i\vec{n}\cdot\vec{t}}$$

where $\lambda_{\vec{n}} := \lambda(e^{-i\vec{n}\cdot\vec{t}})$. Why? Well for each $\lambda \in (H^s(\mathbb{T}^d))^* \ \exists\ v \in H^{-s}(\mathbb{T}^d)$ s.t.

$$\lambda(u) = \langle v, u \rangle = \sum v_{\vec{n}} u_{\vec{n}},$$

so in particular $v = \sum \lambda_{\vec{n}} e^{i\vec{n}\cdot\vec{t}}$.

Therefore, if $\mathcal{D}(\mathbb{T}^d)$ is the space of distributions on $\mathbb{T}^d$, then

$$\mathcal{D}(\mathbb{T}^d) = H^{-\infty}(\mathbb{T}^d).$$

Now, we can define the derivatives of a distribution by

$$D^\alpha \lambda(u) = \lambda(D^\alpha u),$$

so the Fourier series of $D^\alpha\lambda$ is $\sum \lambda_{\vec{n}} \vec{n}^\alpha e^{i\vec{n}\cdot\vec{t}}$.

Notice that in particular a distribution is obtained by taking a finite # of (distributional) derivatives of a continuous function.

Now, a distribution $\lambda$ is in $L^2$ if $\lambda \in H^0(\mathbb{T}^d) \subseteq H^{-\infty}(\mathbb{T}^d)$, so we can describe the Sobolev spaces by: $H^s(\mathbb{T}^d)$ consists of all distributions $\lambda$ s.t. the distributional derivatives $D^\alpha \lambda$ are in $L^2$ $\forall\ |\alpha| \leq s$

Particularly interesting distribution is the Dirac delta function defined by

$$\delta(u) = u(0).$$

Then $\delta$ has formal Fourier series

$$\delta = \sum e^{i\vec{n}\cdot\vec{x}}.$$