Math 8250: Day 3

Def: The tangent bundle of a manifold M^n, denoted TM, is the union of the tangent spaces:

$$TM := \bigcup_{p\in M} T_pM$$

Likewise, if T_p^*M is the cotangent space at p (i.e., the linear functionals on T_pM), then

$$T^*M := \bigcup_{p\in M} T_p^*M$$

is the cotangent bundle of M.

Projection $\pi: TM \to M$ given by $\pi(p,v) = p$

Prop: TM is a $2n$-dimensional mfd.

Proof: Let $\{(U_\alpha, \varphi_\alpha)\}$ be a maximal diff'ble structure on M. Let $(x_1^\alpha, \dots, x_n^\alpha)$ be the coords. of U_α and let

$\{\frac{\partial}{\partial x_1^\alpha}, \dots, \frac{\partial}{\partial x_n^\alpha}\}$ be the associated bases for the tangent spaces of $\varphi_\alpha(U_\alpha)$. Define

$$\begin{aligned} \Psi_\alpha : U_\alpha \times \mathbb{R}^n &\to TM \\ (x_1^\alpha, \dots, x_n^\alpha) &\mapsto \left(\varphi_\alpha(x_1^\alpha, \dots, x_n^\alpha), \sum u_i \frac{\partial}{\partial x_i^\alpha}\right) \end{aligned}$$

Then $\{(U_\alpha \times \mathbb{R}^n, \Psi_\alpha)\}$ gives the differentiable structure on $\mathbb{R}^n$. ▨

Ex:
- $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$
- $TS^1 \cong$ cylinder

Def: Suppose M^m & N^n are differentiable mfds & $\varphi: M \to N$ is smooth. For each $p \in M$ define the differential of f at p, denoted $df_p : T_pM \to T_{f(p)}N$, as follows:

for each $v \in T_pM$, choose a smooth curve $\alpha: (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$. Let $\beta = f \circ \alpha$. Then

$$df_p(v) := \beta'(0).$$

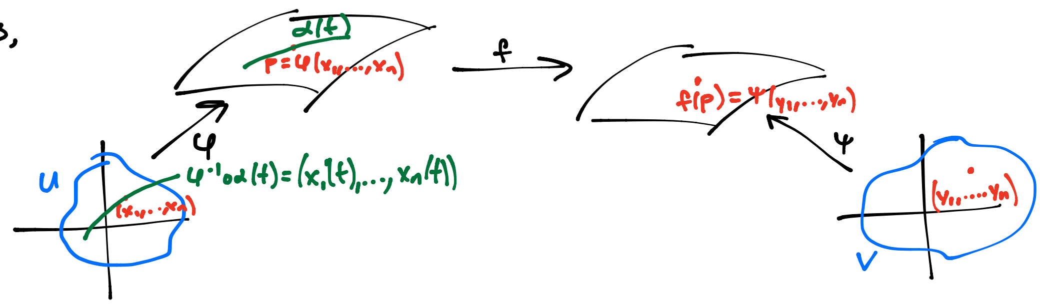
Lemma: This is a well-defined linear map.

Proof: Notice that

$$df_p(v)(g) = \frac{d(g\circ\beta)}{dt}\Big|_{t=0} = \frac{d(g\circ f\circ\alpha)}{dt}\Big|_{t=0} = \frac{d((g\circ f)\circ\alpha)}{dt}\Big|_{t=0} = \alpha'(0)(g\circ f) = v(g\circ f)$$

which is clearly linear & depends only on $\alpha'(0)=v$ & not anything else about α. ∎

In coords,

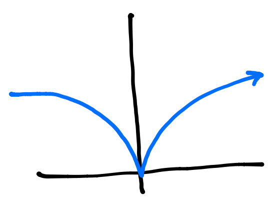


$$(y_1,\dots,y_n)=\psi^{-1}\circ f\circ\varphi(x_1,\dots,x_n)=(y_1(x_1,\dots,x_n),\dots,y_n(x_1,\dots,x_n))$$

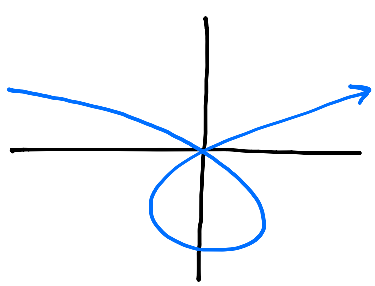
Then $\beta'(0)=\left(\sum \frac{\partial y_1}{\partial x_i}x_i'(0),\dots,\sum\frac{\partial y_n}{\partial x_i}x_i'(0)\right) = \underbrace{\left(\frac{\partial y_i}{\partial x_j}\right)}_{n\times m \text{ matrix}}\left(x_j'(0)\right)$ ← m-element column vector

Def: Suppose $f:M\to N$ is smooth. Then f is an *immersion* if df_p is injective for all $p\in M$. If f is also a homeomorphism onto its image, then f is an *embedding*. The image of an embedding is a *submanifold*. If f is bijective & f^{-1} is smooth, then f is a *diffeomorphism*. f is a *local diffeo.* at $p\in M$ if there exists nbhds U of p & V of $f(p)$ s.t. $f:U\to V$ a diffeo.

Ex: • $\alpha:\mathbb{R}\to\mathbb{R}^2$ given by $\alpha(t)=(t^2,t^3)$ is not an immersion b/c $\alpha'(0)=0$



• $\alpha(t)=(t^3-4t,\ t^2-4)$ is an immersion since $\alpha'(t)\neq 0\ \forall t$, but not an embedding:



Prop: Suppose $f: M \to N$ is smooth. If $p \in M$ & $df_p : T_pM \to T_{f(p)}N$ is an isomorphism, then f is a local diffeo at p.

Pf: Inverse function theorem (on $\psi^{-1} \circ f \circ \varphi : U \to V$)

> IVT: If $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ given by $(x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$, then $\left(\frac{\partial y_i}{\partial x_j}\right) = dF$ non-sing. at $p \Rightarrow F$ a local diffeo. at p.

$\blacksquare$