Math 8250: Day 29

Let $T^d = \underbrace{S^1 \times \dots \times S^1}_{d}$ be the d-torus w/ coords $t_1, \dots, t_d$. Then let $\mathcal{F}(S^1)$ be the space of formal Fourier series

$$u = \sum_{\vec{n} \in \mathbb{Z}^d} c_{\vec{n}} e^{i \vec{n} \cdot \vec{t}}$$

As you've seen, we can formally define the Laplacian by $\Delta u = -\sum_{\vec{n} \in \mathbb{Z}^d} c_{\vec{n}} |\vec{n}|^2 e^{i \vec{n} \cdot \vec{t}}$. Therefore, if

$$\|u\|_{L^2}^2 = \int_{T^d} u^2 \underset{\text{Plancherel identity}}{=} \sum_{\vec{n} \in \mathbb{Z}^d} |c_{\vec{n}}|^2$$

is the usual L^2 inner product on functions on T^d, then

$$\|u\|_{L^2}^2 + \|du\|_{L^2}^2 = \sum_{\vec{n} \in \mathbb{Z}^d} |c_{\vec{n}}|^2 + \sum_{\vec{n} \in \mathbb{Z}^d} |c_{\vec{n}}|^2 |\vec{n}|^2 = \sum (1 + |\vec{n}|^2) |c_{\vec{n}}|^2.$$

In general, taking more derivatives gives more powers of $(1+|\vec{n}|^2)$. If $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$ for some multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$

then for $u \in C^{|\alpha|}(T^d)$,

$$D^\alpha u = \sum_{\vec{n} \in \mathbb{Z}^d} c_{\vec{n}} \vec{n}^\alpha e^{i \vec{n} \cdot \vec{t}}$$

where $\vec{n}^\alpha = n_1^{\alpha_1} \cdots n_d^{\alpha_d}$ and so $\|D^\alpha u\|_{L^2}^2 = \sum |\vec{n}^\alpha|^2 |c_{\vec{n}}|^2$

Therefore,

$$\sum_{|\alpha| \le s} \|D^\alpha u\|_{L^2}^2 = \sum_{|\alpha| \le s} \sum_{\vec{n} \in \mathbb{Z}^d} |\vec{n}^{2\alpha}| |c_{\vec{n}}|^2 = \sum_{\vec{n} \in \mathbb{Z}^d} \left(\sum_{|\alpha| \le s} |\vec{n}^{2\alpha}| \right) |c_{\vec{n}}|^2.$$

Since $\sum_{|\alpha| \le s} |\vec{n}^{2\alpha}| \le (1 + |\vec{n}|^2)^s \le C_s \sum_{|\alpha| \le s} |\vec{n}^{2\alpha}|$, the above norm on $C^s(T^d)$ is equivalent to the Sobolev s-norm:

$$\|u\|_s^2 := \sum_{\vec{n} \in \mathbb{Z}^d} (1 + |\vec{n}|^2)^s |c_{\vec{n}}|^2$$

Of course, the advantage of the s-norm is that it is well-defined on all of $\mathcal{F}(T^d)$, not just the smooth functions.

Even more generally the Sobolev (s,p)-norm is

$$\|u\|_{(s,p)}^p := \sum (1 + |\vec{n}|^p)^s |c_{\vec{n}}|^p$$

which is essentially taking the L^p norm on each derivative.

Def: Let the Sobolev space $W^{s,p}(\mathbb{T}^d)$ to be the Banach space

$$W^{s,p}(\mathbb{T}^d) := \{u \in \mathcal{D}'(\mathbb{T}^d) : \|u\|_{(s,p)} < \infty\}$$

The space $W^{s,2}(\mathbb{T}^d)$ is then also called $H^s(\mathbb{T}^d)$.

Notice, in particular, that $W^{0,p}(\mathbb{T}^d) = L^p(\mathbb{T}^d)$.

Also, $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$ is an inner product space &, hence, a Hilbert space.

Now obviously we have that $W^{s+t,p}(\mathbb{T}^d) \subseteq W^{s,p}(\mathbb{T}^d)$ for all $t \geq 0$.

This lets us define $W^{\infty,p}(\mathbb{T}^d) = \bigcap_s W^{s,p}(\mathbb{T}^d)$ (& in particular $H^{\infty}(\mathbb{T}^d) = W^{\infty,2}(\mathbb{T}^d)$ & $H^{-\infty}(\mathbb{T}^d) = W^{-\infty,2}(\mathbb{T}^d)$)

and $W^{-\infty,p}(\mathbb{T}^d) = \bigcup_s W^{s,p}(\mathbb{T}^d)$

Also, the above discussion implies that $C^s(\mathbb{T}^d) \subset W^{s,p}(\mathbb{T}^d)$; in fact $W^{s,p}(\mathbb{T}^d)$ is the completion of $C^{\infty}(\mathbb{T}^d)$ in the (s,p)-norm.

Remarkably, a partial converse also holds:

Sobolev Lemma: For all $s \geq 0$ & $t > 0$, we have the continuous embedding

$$W^{s+t,p}(\mathbb{T}^d) \hookrightarrow C^s(\mathbb{T}^d) \qquad \text{for all } p > \frac{d}{t}.$$

Proof: I'll just prove the case when $p=2$ and s an integer (the most important case for our purposes), so I want to show that any $u \in H^{s+t}(\mathbb{T}^d)$ w/ $t > d/2$ is the Fourier series of a function in $C^s(\mathbb{T}^d)$ that converges uniformly to the function.

The goal is to prove this by induction on s, so first I'll prove it when $s=0$:

If $u = \sum c_{\vec{n}} e^{i\vec{n}\cdot\vec{t}}$ with $\|u\|_t^2 = \sum_{\vec{n}\in\mathbb{Z}^d} (1+|\vec{n}|^2)^t |c_{\vec{n}}|^2 < \infty$,

with R^{th} partial sum

$$S_R = \sum_{|\vec{n}| \leq R} c_{\vec{n}} e^{i\vec{n}\cdot\vec{t}},$$

then each S_R is continuous & for $R \leq R'$,

$$|S_R(\vec{t}) - S_{R'}(\vec{t})|^2 \le \sum_{|\vec{n}|\ge R} |c_{\vec{n}}|^2 = \sum_{|\vec{n}|\ge R} \frac{(1+|\vec{n}|^2)^t |c_{\vec{n}}|^2}{(1+|\vec{n}|^2)^t}$$

$$\le \|u\|_t^2 \sum_{|\vec{n}|\ge R} \frac{1}{(1+|\vec{n}|^2)^t}$$

But note that $\sum_{\vec{n}} \frac{1}{(1+|\vec{n}|^2)^t} \le 1 + \sum_{\vec{n}\ne\vec{0}} \frac{1}{|\vec{n}|^{2t}}$, which converges since $2t>d$ & hence $\int_{\mathbb{R}^d - B_1(\vec{0})} \frac{1}{|\vec{x}|^{2t}}\, d\vec{x} < \infty$.

Therefore the sequence of partial sums is uniformly Cauchy. A uniformly Cauchy sequence of continuous functions from any topological space to a complete metric space converges uniformly to a continuous function, so we see that $u \in C^0(T^d)$.

Now, for the inductive step, the proof for $d=1$ contains all the essential ideas, so I'll just do that.

Assume $H^{s+t}(S^1) \subset C^s(S^1)$ & suppose $u \in H^{s+t+1}(S^1)$; in particular,

$$u = \sum c_n e^{int} \quad \text{s.t.} \quad \sum_n |n|^{2s+2t+2} |c_n|^2 < \infty.$$

Let $v := \sum_{n\ne 0} in c_n e^{int}$. Then $\sum_n |n|^{2s+2t} |inc_n|^2 = \sum_n |n|^{2s+2t+2} |c_n|^2 < \infty$, so $v \in H^{s+t}(S^1)$

so, by the inductive hypothesis, $v \in C^s(S^1)$. Since convergence of the Fourier series is uniform, we can integrate term-by-term:

$$\int_0^t v(\theta)\,d\theta = \sum_{n\ne 0} \int_0^t in c_n e^{ins}\, ds = \sum_{n\ne 0} c_n e^{int} = u(t) - u(0),$$

so $u'(t) = v(t)$ and so $u \in C^{s+1}(S^1)$, as desired. ∎