Math 8250: Day 28

Hodge-podge of consequences & examples

First off, let's use Hodge to see Poincaré duality as concretely as possible:

Let $\alpha \in H^p_{dR}(M)$ & let $\omega \in \mathcal{H}^p(M)$ be the unique harmonic rep.; i.e., $[\omega] = \alpha \in H^p_{dR}(M)$

Now, we know that $PD(\alpha) \in (H^{n-p}_{dR}(M))^*$ s.t. $PD(\alpha)(\beta) = \mu(\alpha \cup \beta)$ for all $\beta \in H^{n-p}_{dR}(M)$.

Well, we know that if $\eta \in \Omega^{n-p}(M)$ is a rep. of β, then

$$\mu(\alpha \cup \beta) = \int_M \omega \wedge * * \eta = \pm \langle * \omega, \eta \rangle.$$

Now, since $\omega \in \mathcal{H}^p(M)$ & since $*$ commutes w/ Δ (this is a calculation), it follows that

$$\Delta * \omega = * \Delta \omega = 0,$$

so $*\omega \in \mathcal{H}^{n-p}(M)$. In particular, $*\omega$ is the unique harmonic rep. of the deRham class $[*\omega] \in H^{n-p}_{dR}(M)$.

So in this sense Poincaré duality is very explicitly realized by $*$ acting on harmonic forms.

Claim: The unique harmonic representative ω of a de Rham cohomology class in $H^p_{dR}(M)$ has the minimum L^2 norm among all representatives of the class.

Indeed, if ω' is cohomologous to ω, then $\omega' = \omega + d\gamma$ for some $\gamma \in \Omega^{p-1}(M)$. But then

$$\|\omega'\|^2 = \|\omega + d\gamma\|^2 = \langle \omega + d\gamma, \omega + d\gamma \rangle = \langle \omega, \omega \rangle + \underbrace{\cancel{2\langle \omega, d\gamma \rangle}}_{0 \text{ since } \mathcal{H}^p(M) \perp d\Omega^{p-1}(M)} + \langle d\gamma, d\gamma \rangle$$

$$= \|\omega\|^2 + \|d\gamma\|^2 \geq \|\omega\|^2, \text{ with equality} \Leftrightarrow d\gamma = 0.$$

If we think of the L^2 norm as an energy (as we so often do), then the harmonic rep. of a cohomology class minimizes energy among all reps of that class.

Now, I want to spend some time working out essentially everything we've learned this semester on the 3-torus:

Note, first of all, that a C^∞ fcn $f: S^1 \to \mathbb{R}$ can be expanded as a Fourier series

$$f(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

Since f is real-valued, $\overline{c_n} = c_{-n}$. Note that $f'(\theta) = \sum_{n\in\mathbb{Z}} inc_n e^{in\theta}$, so

$$df = \sum_{n\in\mathbb{Z}} inc_n e^{in\theta}\, d\theta$$

In turn, $\delta = -\star d \star$, so for a 1-form $\alpha = \sum c_n e^{in\theta} d\theta$,

$$\delta\alpha = -\star d\star\alpha = -\star d \sum c_n e^{in\theta} = -\star \sum inc_n e^{in\theta} d\theta = -\sum inc_n e^{in\theta}.$$

Note, in particular, that $d\Omega^0(S^1) = \{\sum c_n e^{in\theta} d\theta : c_0 = 0\}$ & $\delta\Omega^1(S^1) = \{\sum c_n e^{in\theta} : c_n = 0\}$.

Hence, $H^1_{dR}(S^1) = \Omega^1(S^1) / d\Omega^0(S^1) \cong \{c_0\}$, which is 1-dim'l.

Now, if $\alpha = \sum c_n e^{in\theta}$, then $\Delta\alpha = (d\delta + \delta d)\alpha = d\delta\alpha = \sum c_n n^2 e^{in\theta} d\theta$ and so

$$\mathcal{H}^1(S^1) = \{\sum c_n e^{in\theta} d\theta : \sum c_n n^2 e^{in\theta} d\theta = 0\} = \{c_0\}.$$

Hence, the "constant" 1-forms are the only harmonic 1-forms (& likewise for 0-forms).

On $T^3 = S^1 \times S^1 \times S^1$, we again write everything out as (3-variable) Fourier series:

- If $f \in \Omega^0(T^3)$, then $f(s,t,u) = \sum_{\vec{n}\in\mathbb{Z}^3} c_{\vec{n}} e^{i\vec{n}\cdot\vec{x}}$, where $\vec{x} = (s,t,u)$.

- If $\alpha \in \Omega^1(S^1)$, then $\alpha = \left(\sum_{\vec{n}\in\mathbb{Z}^3} c^{(1)}_{\vec{n}} e^{i\vec{n}\cdot\vec{x}}\right) ds + \left(\sum_{\vec{n}\in\mathbb{Z}^3} c^{(2)}_{\vec{n}} e^{i\vec{n}\cdot\vec{x}}\right) dt + \left(\sum_{\vec{n}\in\mathbb{Z}^3} c^{(3)}_{\vec{n}} e^{i\vec{n}\cdot\vec{x}}\right) du$

$$= \sum_{\vec{n}\in\mathbb{Z}^3} \vec{c}_{\vec{n}} e^{i\vec{n}\cdot\vec{x}} \cdot d\vec{x}$$

where $\vec{c}_{\vec{n}} = (c^{(1)}_{\vec{n}}, c^{(2)}_{\vec{n}}, c^{(3)}_{\vec{n}})$, $d\vec{x} = (ds, dt, du)$ &

$$\vec{c}_{\vec{n}} \cdot d\vec{x} = c^{(1)}_{\vec{n}} ds + c^{(2)}_{\vec{n}} dt + c^{(3)}_{\vec{n}} du.$$

- If $\beta \in \Omega^2(T^3)$, then $\beta = \left(\sum c_{\vec n}^{(1)} e^{i\vec n\cdot\vec x}\right) dt\wedge du + \left(\sum c_{\vec n}^{(2)} e^{i\vec n\cdot\vec x}\right) du\wedge ds + \left(\sum c_{\vec n}^{(3)} e^{i\vec n\cdot\vec x}\right) ds\wedge dt$

$$= \sum \vec c_{\vec n}\, e^{i\vec n\cdot\vec x}\cdot {\star d\vec x}$$

where $\star d\vec x = (dt\wedge du,\, du\wedge ds,\, ds\wedge dt)$.

- If $\gamma \in \Omega^3(T^3)$, then $\gamma = \sum c_{\vec n} e^{i\vec n\cdot\vec x}\, dVol$

where $dVol = ds\wedge dt\wedge du$.

Now, what is the effect of d?

- For $f \in \Omega^0(T^3)$, $df = f_s\,ds + f_t\,dt + f_u\,du$

$$= \sum_{\vec n\in\mathbb{Z}^3} c_{\vec n}\, i\, e^{i\vec n\cdot\vec x}(n_1\,ds + n_2\,dt + n_3\,du)$$

$$= \sum_{\vec n\in\mathbb{Z}^3} c_{\vec n}\, i\,\vec n\, e^{i\vec n\cdot\vec x}\cdot d\vec x$$

- For $\alpha \in \Omega^1(T^3)$, $d\alpha = \sum c_{\vec n}^{(1)} i(n_2\,dt + n_3\,du) e^{i\vec n\cdot\vec x}\wedge ds + \sum c_{\vec n}^{(2)} i(n_3\,du + n_1\,ds) e^{i\vec n\cdot\vec x}\wedge dt$

$$+ \sum c_{\vec n}^{(3)} i(n_1\,ds + n_2\,dt) e^{i\vec n\cdot\vec x}\wedge du$$

$$= \sum \left(c_{\vec n}^{(3)} n_2 - c_{\vec n}^{(2)} n_3\right)(dt\wedge du)\, i e^{i\vec n\cdot\vec x} + \sum\left(c_{\vec n}^{(1)} n_3 - c_{\vec n}^{(3)} n_1\right)(du\wedge ds)\, i e^{i\vec n\cdot\vec x}$$

$$+ \sum\left(c_{\vec n}^{(2)} n_1 - c_{\vec n}^{(1)} n_2\right)(ds\wedge dt)\, i e^{i\vec n\cdot\vec x}$$

$$= \sum (\vec n\times\vec c_{\vec n})\, i e^{i\vec n\cdot\vec x}\cdot {\star dx}$$

- For $\beta\in\Omega^2(T^3)$, $d\beta = \dots = \sum \vec c_{\vec n}\cdot\vec n\, i e^{i\vec n\cdot\vec x}\cdot {\star d\vec x}$

- For $\gamma\in\Omega^3(T^3)$, $d\gamma = 0$

So, for example, we can see that $\alpha\in\Omega^1(T^3)$ · is in the kernel of $d \iff \vec c_{\vec n}$ is parallel to $\vec n$ $\forall \vec n\neq\vec 0$

· is in image of $d \iff \vec c_{\vec n} \parallel \vec n\ \forall\vec n\neq\vec 0\ \&\ \vec c_{\vec 0} = \vec 0$

So, in particular, $H^1_{dR}(T^3) = \ker d/\operatorname{im} d = \dfrac{\{\vec c_{\vec n}\parallel\vec n\ \forall\vec n\neq\vec 0\}}{\{\vec c_{\vec n}\parallel\vec n\ \forall\vec n\neq\vec 0\ \&\ \vec c_{\vec 0}=\vec 0\}} = \{\vec c_{\vec 0}\} = \mathbb{R}^3.$