Math 8250: Day 27

Remember that proving the Hodge Thm basically boils down to showing that the PDE

$$\Delta\omega=\alpha$$

has a solution $\Leftrightarrow \alpha\perp\mathcal{H}^p(M)$.

So we really need to get a good handle on solutions to this equation.

The basic strategy is to: ① find a weak solution

and ② Show that any weak solution is actually a solution.

Now, recall that Δ is self-adjoint, so any solution of $\Delta\omega=\alpha$ will satisfy

$$\langle\alpha,\varphi\rangle=\langle\Delta\omega,\varphi\rangle=\langle\omega,\Delta\varphi\rangle$$

for any $\varphi\in\Omega^p(M)$. Therefore, any actual solution ω defines an operator $\ell:\Omega^p(M)\to\mathbb{R}$ by $\ell(\eta)=\langle\omega,\eta\rangle$

s.t. $\ell(\Delta\varphi)=\langle\alpha,\varphi\rangle$

This motivates the following definition:

Def: A weak solution of the equation $\Delta\omega=\alpha$ is a bounded linear map $\ell:\Omega^p(M)\to\mathbb{R}$ s.t. $\ell(\Delta\varphi)=\langle\alpha,\varphi\rangle$ for every $\varphi\in\Omega^p(M)$.

Then the key to proving Hodge will be showing:

Regularity Thm: Let $\alpha\in\Omega^p(M)$ & let $\ell\in\Omega^p(M)^*$ be a weak solution of $\Delta\omega=\alpha$. Then $\exists\,\omega\in\Omega^p(M)$

so that $\ell(\varphi)=\langle\omega,\varphi\rangle$ for all $\varphi\in\Omega^p(M)$. Therefore $\Delta\omega=\alpha$.

The last part is easy: if any weak solution ℓ is given by $\ell(\varphi)=\langle\omega,\varphi\rangle$, then

$$\langle\alpha,\varphi\rangle=\ell(\Delta\varphi)=\langle\omega,\Delta\varphi\rangle=\langle\Delta\omega,\varphi\rangle$$

for all φ, and so $\Delta\omega=\alpha$.

The other key point is:

Thm: If $\{\varphi_n\}$ is a seq. of smooth p-forms & $\|\varphi_n\|\le c$ and $\|\Delta\varphi_n\|\le c$ $\forall n$ and some $c>0$, then a subsequence of $\{\varphi_n\}$ is Cauchy.

Proof of Hodge: First, to see that $\mathcal{H}^p(M)$ is finite-dim'l, suppose it wasn't. Then there would be an infinite ON sequence of harmonic p-forms $\{\varphi_k\}$. But then $\|\varphi_k\|=1$ & $\|\Delta\varphi_k\|=0$ $\forall k$, so the above Thm implies that there is a Cauchy subsequence. But since the φ_k are pairwise ON, this is impossible. So conclude that $\mathcal{H}^p(M)$ is finite-dim'l.

Now, we need to show that $(\mathcal{H}^p(M))^\perp=\Delta\Omega^p(M)$. The containment $\Delta\Omega^p(M)\subset(\mathcal{H}^p(M))^\perp$ is obvious (why?)

OTOH, suppose $\alpha\in(\mathcal{H}^p(M))^\perp$. Then define $\ell:\Delta\Omega^p(M)\to\mathbb{R}$ by

$$\Delta\varphi\mapsto\langle\alpha,\varphi\rangle.$$

Certainly, if $\Delta\varphi=0$ then

$$\langle\alpha,\varphi\rangle=0$$

since $\alpha\in(\mathcal{H}^p(M))^\perp$, so ℓ is well-defined. To see that ℓ is bounded, we will need the following:

Lemma: $\exists$ a uniform constant $c>0$ s.t. $\|\varphi\|\le c\|\Delta\varphi\|$ $\forall$ $\varphi\in(\mathcal{H}^p(M))^\perp$

With the lemma in hand we have that

$$|\ell(\Delta\varphi)|=|\langle\alpha,\varphi\rangle|=|\langle\alpha,\varphi-H(\varphi)\rangle|\le\|\alpha\|\,\|\varphi-H(\varphi)\|\le c\|\alpha\|\,\|\Delta(\varphi-H(\varphi))\|=c\|\alpha\|\,\|\Delta\varphi\|$$

Then Hahn-Banach gives us that ℓ extends to a bdd linear fnl on $\Omega^p(M)$, so ℓ is a weak solution of $\Delta\omega=\alpha$.

But then the regularity theorem implies that there is a strong solution ω s.t. $\Delta\omega=\alpha$. But then that means $(\mathcal{H}^p(M))^\perp\subseteq\Delta\Omega^p(M)$.

Proof of Lemma: Suppose not. Then I can find a sequence $\{\varphi_k\}$ s.t. $\|\varphi_k\|=1$ & $\|\Delta\varphi_k\|\to0$. Then by the theorem I know that there is a Cauchy subsequence $\{\varphi_{k_j}\}$. But then

$$\ell(\xi):=\lim_{j\to\infty}\langle\varphi_{k_j},\xi\rangle$$

is a well-defined linear operator on $\Omega^p(M)$. Then l is bdd &

$$l(\Delta\psi) = \lim_{j\to\infty} \langle \varphi_{k_j}, \Delta\psi \rangle = \lim_{j\to\infty} \langle \Delta\varphi_{k_j}, \psi \rangle = 0$$

for all $\psi \in \Omega^p(M)$, so l is a weak soln of $\Delta\omega = 0$. But then by the regularity theorem $\exists\ \omega$ s.t.

$$\langle \omega, \xi \rangle = l(\xi) = \lim_{j\to\infty} \langle \varphi_{k_j}, \xi \rangle,$$

so $\varphi_{k_j} \to \omega$. But clearly $\{\varphi_{k_j}\}$ is a Cauchy seq. in $(\mathcal{H}^p(M))^\perp$ & $\omega \in \mathcal{H}^p(M)$ w/ $\|\omega\| = 1$. This is obviously a contradiction. ∎