Math 8250: Day 26

One immediate consequence of the definition is

Prop: Δ is self-adjoint.

Proof: Clearly, if $\alpha, \beta \in \Omega^p(M)$,

$$\langle \Delta\alpha, \beta\rangle = \langle d\delta\alpha + \delta d\alpha, \beta\rangle = \underbrace{\langle \delta\alpha, \delta\beta\rangle + \langle d\alpha, d\beta\rangle}_{\text{called the "Dirichlet integral" of } \alpha \,\&\, \beta} = \langle \alpha, d\delta\beta\rangle + \langle \alpha, \delta d\beta\rangle = \langle \alpha, \Delta\beta\rangle. \quad \blacksquare$$

In fact, the same proof shows that for any operator defined as $LL^* + L^*L$ for some operator L is self-adjoint.

Now, we will be very interested in the harmonic forms on a closed mfd:

Def: Let M^n be a cpct, oriented, Riemannian mfd. Define the space

$$\mathcal{H}^p(M) := \{\omega \in \Omega^p(M) \mid \Delta\omega = 0\}$$

of harmonic p-forms on M.

Prop: When M is closed, $\mathcal{H}^p(M)$ is precisely $\ker d \cap \ker \delta$, the space of closed & co-closed p-forms.

Proof: If $\omega \in \ker d \cap \ker \delta$, then

$$\Delta\omega = (d\delta + \delta d)\omega = d\delta\omega + \delta d\omega = 0.$$

OTOH, if $\omega \in \ker \Delta$, then

$$0 = \langle \Delta\omega, \omega\rangle = \langle d\delta\omega, \omega\rangle + \langle \delta d\omega, \omega\rangle = \langle \delta\omega, \delta\omega\rangle + \langle d\omega, d\omega\rangle = \|\delta\omega\|^2 + \|d\omega\|^2,$$

which can only happen if $d\omega = 0$ & $\delta\omega = 0$. $\blacksquare$

Note: When M has boundary, this is no longer true. We will only have that $\ker d \cap \ker \delta \subsetneq \mathcal{H}^p(M)$. The space $\ker d \cap \ker \delta$ actually turns out to be more important. To distinguish it, Kodaira called it the space of harmonic p-fields

Hodge Decomposition Theorem: For a closed, oriented mfd M^n and for any $0 \le p \le n$, the space of harmonic p-forms $\mathcal{H}^p(M)$ is finite-dimensional. Moreover, we have the following orthogonal direct sum decompositions of the space of p-forms:

$$\begin{aligned}\Omega^p(M) &= \Delta\Omega^p(M) \oplus \mathcal{H}^p(M)\\ &= d\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M) \oplus \mathcal{H}^p(M).\end{aligned}$$

Proving this will be a major undertaking, but here's the executive summary:

(1) If we can show that $\mathcal{H}^p(M)$ is finite-dimensional, then it automatically follows that

$$\Omega^p(M) = (\mathcal{H}^p(M))^\perp \oplus \mathcal{H}^p(M).$$

(2) Then the hard part is to show that $(\mathcal{H}^p(M))^\perp = \Delta\Omega^p(M)$. One direction is obvious, since, for $\alpha \in \Delta\Omega^p(M)$ & $\omega \in \mathcal{H}^p(M)$, $\langle \Delta\alpha, \omega\rangle = \langle \alpha, \Delta\omega\rangle = 0$ since Δ is self-adjoint. The hard part is to show that $(\mathcal{H}^p(M))^\perp \subset \Delta\mathcal{H}^p(M)$. In other words, if $\alpha \perp \mathcal{H}^p(M)$, then the PDE

$$\Delta\omega = \alpha$$

has a (unique) solution.

The major consequence of the Hodge decomposition is:

Thm: Each de Rham cohomology class on a closed, oriented manifold has a unique harmonic representative.

Cor: The cohomology groups of a closed mfd are all finite-dimensional.

Def: For each $\alpha \in \Omega^p(M)$, let $H(\alpha)$ be the projection of α onto $\mathcal{H}^p(M)$. $H(\alpha)$ is called the *harmonic part* of α.

Def: For each p, define the *Green's operator* $G: \Omega^p(M) \to \Omega^p(M)$ as follows: for each $\alpha \in \Omega^p(M)$, let $G\alpha$ be the unique solution of $\Delta\omega = \alpha - H(\alpha)$.

Lemma: If T is an operator that commutes w/ Δ, then G commutes w/ T.

Ex: G commutes w/ d: $G\alpha$ is the soln of

$$\Delta\omega = \alpha - H(\alpha).$$

But then $\Delta dG\alpha = d\Delta G\alpha = d(\alpha - H(\alpha)) = d\alpha = d\alpha - H(d\alpha)$,

so $dG\alpha = Gd\alpha$.

Pf of Thm: For any $\alpha \in \Omega^p(M)$,

$$\alpha = \Delta G\alpha + H(\alpha) = d\delta G\alpha + \delta dG\alpha + H(\alpha) = d\delta G\alpha + \delta Gd\alpha + H(\alpha).$$

But then if α is closed (& thus reps a de Rham class) then the middle term is zero, & so

$$\alpha = \text{exact} + \text{harmonic}.$$

Hence, any cohomology class has a harmonic representative. To show it is unique, suppose $\omega_1, \omega_2 \in \mathcal{H}^p(M)$ represent the same class. Then $\omega_1 - \omega_2 = d\gamma$ for some $\gamma \in \Omega^{p-1}(M)$.

But now $\omega_1 - \omega_2 \in \mathcal{H}^p(M)$ & we know from the Hodge Thm that $\mathcal{H}^p(M) \perp d\Omega^{p-1}(M)$. Therefore,

$$\langle \omega_1 - \omega_2, \omega_1 - \omega_2 \rangle = \langle \omega_1 - \omega_2, d\gamma \rangle = 0,$$

so $\omega_1 = \omega_2$. ∎