Math 8250: Day 25

Suppose M^n is closed & oriented. Define the **cup product** $\cup: H^p_{dR}(M) \times H^q_{dR}(M) \to H^{p+q}_{dR}(M)$ by

$$[\omega] \cup [\eta] = [\omega \wedge \eta].$$

① $\cup$ is well-defined:
- Clearly, $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^p \omega \wedge d\eta$, so $\omega\wedge\eta$ closed if ω & η are.
- If $\omega = d\alpha$, then $\omega\wedge\eta = d\alpha\wedge\eta = d(\alpha\wedge\eta) + (-1)^{p-1}\alpha\wedge d\eta = d(\alpha\wedge\eta)$.

  Likewise if η is exact, so $\omega\wedge\eta$ is exact if either ω or η is exact.

② $\cup$ is bilinear: obvious

③ For $\alpha\in H^p_{dR}(M)$, $\beta\in H^q_{dR}(M)$, $\alpha\cup\beta = (-1)^{pq}\beta\cup\alpha$. Also obvious since true of wedge product.

Now, suppose M is connected & that the orientation is given by $\mu = [\eta] \in H^n_{dR}(M)$, where $\int_M \eta = 1$.

By abuse of notation, I will also use μ to denote the isomorphism $H^n_{dR}(M) \to \mathbb{R}$ that sends this element to 1.

In other words, for any $[\tau] \in H^n_{dR}(M)$, $\mu([\tau]) = \int_M \tau$.

Now, for each $0 \le p \le n$ we define the map $PD: H^p_{dR}(M) \to H^{n-p}_{dR}(M)^*$ by

$$\alpha \longmapsto (\beta \longmapsto \mu(\alpha\cup\beta))$$

In other words, $PD(\alpha)(\beta) = \mu(\alpha\cup\beta)$.

Thm (Poincaré duality): If M^n is connected, oriented, closed mfld, then the map

$$PD: H^p_{dR}(M) \to H^{n-p}_{dR}(M)^*$$

is an isomorphism for all p.

Corollary: $H^p_{dR}(M) \cong H^{n-p}_{dR}(M)$

Proof: As we will eventually show, $H^k_{dR}(M)$ is finite-dim'l whenever M is compact. Since $V \cong V^*$ for any finite-dim'l v.s. V, we have $H^p_{dR}(M) \underset{P.D.}{\cong} H^{n-p}_{dR}(M)^* \cong H^{n-p}_{dR}(M)$. ∎

Corolly: If M is odd-dimensional, $\chi(M)=0$.

Proof: By def & the iso. btw de Rham & singular cohomology,

$$\chi(M)=\sum_{i=0}^{n}(-1)^i \dim H^i(M;\mathbb{R})=\sum_{i=0}^{n}(-1)^i \dim H^i_{dR}(M).$$

But by prev. corolly these terms cancel in pairs. ∎

We prove the thm in the same way as we proved the isomorphism btw de Rham & singular cohomology.

Specifically, we take a cover of the mfd by a finite # of geodesically convex open sets, then iteratively apply the following lemma:

Lemma: If $M=U\cup V$ for open $U,V\subset M$ s.t. PD is an iso. for U, V, & $U\cap V$, then PD is an iso. for M.

Proof: Consider the diagram below, where the top row is Mayer-Vietoris & the bottom row is the dual of Mayer-Vietoris

$$\begin{tikzcd}
H^{p-1}_{dR}(U)\oplus H^{p-1}_{dR}(V) \arrow[r] \arrow[d, "PD\oplus PD"] & H^{p-1}_{dR}(U\cap V) \arrow[r] \arrow[d, "PD"] & H^p_{dR}(U\cup V) \arrow[r] \arrow[d, "PD"] & H^p_{dR}(U)\oplus H^p_{dR}(V) \arrow[r] \arrow[d, "PD\oplus PD"] & H^p_{dR}(U\cap V) \arrow[d, "PD"] \\
\left[H^{n-p+1}_{dR}(U)\oplus H^{n-p+1}_{dR}(V)\right]^* \arrow[r] & H^{n-p+1}_{dR}(U\cap V)^* \arrow[r] & H^{n-p}_{dR}(U\cup V)^* \arrow[r] & \left[H^{n-p}_{dR}(U)\oplus H^{n-p}_{dR}(V)\right]^* \arrow[r] & H^{n-p}_{dR}(U\cap V)
\end{tikzcd}$$

By assumption all vertical maps except the middle one are isomorphisms. The diagram almost commutes; we have to introduce a sign of $(-1)^p$ in the middle square to make it commute. After doing so, the lemma is a consequence of the Five Lemma. ∎

If M is a Riemannian mfld, then the Poincaré duality map corresponds in a very precise sense to the Hodge star.

Specifically, if $[\omega]\in H^p_{dR}(M)$, then, for any $[\eta]\in H^{n-p}_{dR}(M)$,

$$\begin{aligned}
PD([\omega])([\eta]) &= \mu([\omega]\cup[\eta])=\mu([\omega\wedge\eta])=\int_M \omega\wedge\eta=(-1)^{p(n-p)}\int_M \eta\wedge\omega \\
&=(-1)^{p(n-p)}(-1)^{p(n-p)}\int_M \eta\wedge{*}{*}\omega \\
&=\langle \eta, *\omega\rangle=\langle *\omega,\eta\rangle.
\end{aligned}$$

Now, the inner product isn't exactly defined at the level of cohomology, so you have to take this with a little bit of a grain of salt.

Nonetheless, as we'll see soon enough, when we take a harmonic representative of a cohomology class, its Poincaré dual really will be given just by applying the Hodge star.

Okay, we're now in position to start talking about the Hodge theorem.

The first order of business is to notice that on closed Riemannian manifolds, we can very explicitly write down the adjoint of the exterior derivative $d: \Omega^{p-1}(M) \to \Omega^p(M)$:

For $\alpha \in \Omega^{p-1}(M)$ & $\beta \in \Omega^p(M)$,

$$\langle d\alpha, \beta \rangle = \int_M d\alpha \wedge \star\beta = \int_M \left(d(\alpha\wedge\star\beta) - (-1)^{p-1}\alpha\wedge d\star\beta\right) = \overbrace{\cancel{\int_M d(\alpha\wedge\star\beta)}}^{0 \text{ by Stokes}} + (-1)^p \int_M \alpha \wedge d\star\beta$$

$$= (-1)^p\,(-1)^{(p-1)(n-p+1)}\int_M \alpha\wedge\star\star d\star\beta$$

$$= (-1)^{pn-n-1}\langle \alpha, \star d\star\beta\rangle$$

$$= \langle \alpha, (-1)^{pn-n-1}\star d\star\beta\rangle.$$

So the adjoint of d is $\delta := (-1)^{pn-n-1}\star d\star : \Omega^p(M) \to \Omega^{p-1}(M)$, which we call the exterior codifferential.

So we get a dual de Rham complex

$$\Omega^n(M) \xrightarrow{\delta} \Omega^{n-1}(M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^2(M) \xrightarrow{\delta} \Omega^1(M) \xrightarrow{\delta} \Omega^0(M)$$

which is a chain complex since, for $\omega \in \Omega^p(M)$,

$$\delta\delta\omega = \pm\star d\star\star d\star\omega = \pm\star dd\star\omega = 0.$$

Now, we can define the Laplace-Beltrami operator $\Delta : \Omega^p(M) \to \Omega^p(M)$ by

$$\Delta = (d+\delta)^2 = d\delta + \delta d.$$

Ex: Suppose f is a 0-form (i.e., smooth function) on $\mathbb{R}^n$. Then

$$\Delta f = -\sum \frac{\partial^2 f}{\partial x_i^2}, \qquad \text{(HW)}$$

which is minus the usual function Laplacian (this sign choice is called the geometer's sign convention).

Ex: Suppose $\vec{F}$ is a vector field on a domain $M \subset \mathbb{R}^3$. Then recall that the vector Laplacian is

$$\Delta \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times \nabla \times \vec{F}$$

But then recall the diagram

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "\nabla"] \arrow[d, "id"] & \mathfrak{X}(M) \arrow[r, "\nabla\times"] \arrow[d, "\flat"] & \mathfrak{X}(M) \arrow[r, "\nabla\cdot"] \arrow[d, "*\circ\flat"] & C^\infty(M) \arrow[d, "*"] \\
\Omega^0(M) \arrow[r, shift left, "d"] & \Omega^1(M) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Omega^2(M) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Omega^3(M) \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

Then we can think of $\nabla\cdot$ as corresponding to either $d : \Omega^2(M) \to \Omega^3(M)$ or to $\delta : \Omega^1(M) \to \Omega^0(M)$

Likewise, $\nabla\times$ corresponds to both $d : \Omega^1(M) \to \Omega^2(M)$ and to $\delta : \Omega^2(M) \to \Omega^1(M)$

Finally, ∇ can be either $d : \Omega^0(M) \to \Omega^1(M)$ or $\delta : \Omega^3(M) \to \Omega^2(M)$.

But then clearly $\Delta = \nabla(\nabla\cdot) - \nabla\times\nabla\times$ corresponds to $-(d\delta + \delta d) : \Omega^1(M) \to \Omega^1(M)$.