Math 8250: Day 24

HW 5, #3: Most people said that a volume form can't integrate to zero, but why?

HW 6, #4: Z, Z_1, & Z_2 are supposed to be closed, not just compact.

Recall...

de Rham Theorem: There exists a linear map

$$H^p_{dR}(M) \to H_p(M;\mathbb{R})^*$$

which is a vector space isomorphism.

The proof is only half a step up from what you're doing in HW 6 #4.

Specifically, remember that for each p, $C_p(M;\mathbb{R}) = \{\text{smooth singular } p\text{-chains on } M\} = \left\{\sum_{i=1}^k a_i\sigma_i \,\middle|\, a_i \in \mathbb{R}, k \in \mathbb{N}, \sigma_i \text{ a smooth singular } p\text{-simplex}\right\}$

And we have the chain complex

$$\cdots \xrightarrow{\partial_{p+1}} C_p(M;\mathbb{R}) \xrightarrow{\partial_p} C_{p-1}(M;\mathbb{R}) \xrightarrow{\partial_{p-1}} \cdots$$

Then $H_p(M;\mathbb{R}) := \ker\partial_p / \operatorname{im}\partial_{p+1}$. A class $[z] \in H_p(M;\mathbb{R})$ can be represented by a p-cycle z, which is to say, a p-chain $z = \sum_{i=1}^k a_i\sigma_i$ s.t. $\partial z = 0$.

Now, we can define integration over p-chains as follows:

Since $\int$ is linear, for any p-form $\omega \in \Omega^p(M)$ and any $c \in C_p(M;\mathbb{R})$ w/ $c = \sum a_i\sigma_i$, we define

$$\int_c \omega := \sum a_i \int_{\sigma_i(\Delta^p)} \omega = \sum a_i \int_{\mathbb{R}^p} \sigma_i^*\omega,$$

which is perfectly sensible since each σ_i is a smooth map from a nbhd of $\Delta^p \subset \mathbb{R}^p$ to M.

Now, by exactly the same reasoning as in the HW problem:

① If $z \in C_p(M;\mathbb{R})$ is a cycle & ω_1, ω_2 are closed p-forms representing the same cohomology class, then $\int_z \omega_1 = \int_z \omega_2$

② As a result, $\int_z : H^p_{dR}(M) \to \mathbb{R}$ is a linear map s.t.

③ If $z = \partial c$ for $c \in C_{p+1}(M;\mathbb{R})$, then $\int_z = \int_{\partial c}$ is the zero map.

④ If $z_1 - z_2 = \partial c$, then $\int_{z_1} = \int_{z_2}$.

Therefore, the map $\varphi : H_p(M;\mathbb{R}) \to H^p_{dR}(M)^*$ given by

$$z \longmapsto \left(\omega \mapsto \int_z \omega\right)$$

is a linear map. Equivalently, have a map $H^p_{dR}(M) \to H_p(M;\mathbb{R})^* = H^p(M;\mathbb{R})$.

Then de Rham's theorem will follow if we can show that it is an isomorphism. We go in stages

① If M is contractible, $H_p(M;\mathbb{R}) = \begin{cases} 0 & p>0 \\ \mathbb{R} & p=0 \end{cases}$ & $H^p_{dR}(M) = \begin{cases} 0 & p>0 \\ \mathbb{R} & p=0 \end{cases}$, so

only need to check that the map is an iso. for $p=0$. But then a 0-cycle z is a finite linear combination of points, and integrating a function (i.e. 0-form) over a point is just evaluating the function at the point.

Since the only closed 0-forms are constant functions, we see that for nonzero $f = c \in \Omega^0(M)$ & $z \in M$, $\int_z f = f(z) = c \neq 0$, so the map has trivial kernel.

② If $M = U \cup V$ where U & V are contractible open sets (i.e. diffeo. to the open unit disk in $\mathbb{R}^n$) and φ is known to be an iso. for $U \cap V$:

By Mayer-Vietoris,

$$\to H^{p-1}_{dR}(U\cap V) \xrightarrow{\delta^\#} H^p_{dR}(U\cup V) \longrightarrow \underset{0}{H^p_{dR}(U)} \oplus \underset{0}{H^p_{dR}(V)} \longrightarrow H^p_{dR}(U\cap V) \xrightarrow{\delta^\#} H^{p+1}_{dR}(U\cup V) \to \dots$$

so $\delta^\# : H^p_{dR}(U\cap V) \to H^{p+1}_{dR}(U\cup V)$ is an iso. for each $p>0$.

Likewise, by Mayer-Vietoris for singular cohomology, $\tilde{\delta} : H_{p+1}(U\cup V;\mathbb{R}) \to H_p(U\cap V;\mathbb{R})$ is an iso.

Then we have

$$\begin{tikzcd}
\cdots \arrow[r] & \cancel{H^p_{dR}(U)\oplus H^p_{dR}(V)}\ 0 \arrow[r] \arrow[d] & H^p_{dR}(U\cap V) \arrow[r] \arrow[d] & H^{p+1}_{dR}(U\cup V) \arrow[r] \arrow[d] & \cancel{H^{p+1}_{dR}(U)\oplus H^{p+1}_{dR}(V)}\ 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \cancel{H^p(U;\mathbb{R})\oplus H^p(V;\mathbb{R})}\ 0 \arrow[r] & H^p(U\cap V;\mathbb{R}) \arrow[r] & H^{p+1}(U\cup V;\mathbb{R}) \arrow[r] & \cancel{H^{p+1}(U;\mathbb{R})\oplus H^{p+1}(V;\mathbb{R})}\ 0 \arrow[r] & \cdots
\end{tikzcd}$$

Since the map $H^p_{dR}(U\cap V)\to H^p(U\cap V;\mathbb{R})$ is an iso. and the rows are exact, so is the map $H^{p+1}_{dR}(U\cup V)\to H^{p+1}(U\cup V;\mathbb{R})$

The proof is a diagram chase. Alternatively, you can view it as a (very simple) special case of the

<u>Five lemma:</u> Suppose the rows of the following commutative diagram of vector spaces are exact

$$\begin{tikzcd}
V_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"'] & V_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"'] & V_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"'] & V_4 \arrow[r, "\alpha_4"] \arrow[d, "\varphi_4"'] & V_5 \arrow[d, "\varphi_5"'] \\
W_1 \arrow[r, "\beta_1"] & W_2 \arrow[r, "\beta_2"] & W_3 \arrow[r, "\beta_3"] & W_4 \arrow[r, "\beta_5"] & W_5
\end{tikzcd}$$

and that $\varphi_1,\varphi_2,\varphi_4,\varphi_5$ are isomorphisms. Then φ_3 is also an isomorphism.

Again, the proof is a diagram chase.

③ An arbitrary compact mfd can be covered by a finite collection of geodesically convex open sets $U_1,\dots,U_k$

Then the deRham theorem is true for any geodesically convex set, since such a set is contractible.

Moreover, the finite intersection of convex sets is convex, so the deRham theorem is also true for any $U_{i_1}\cap\dots\cap U_{i_\ell}$.

Then by iterative application of ②, the deRham theorem is true for the mfd as well.

Next time: Cup products & Poincaré duality.