Math 8250: Day 23

The calculation of the de Rham cohomology of S^n that we did last time is an example of the following:

Mayer-Vietoris Sequence for de Rham Cohomology: Suppose A, B are smooth, oriented n-manifolds. Then there exists a long exact sequence

$$0 \to H^0_{dR}(A\cup B) \longrightarrow H^0_{dR}(A)\oplus H^0_{dR}(B) \to H^0_{dR}(A\cap B) \to H^1_{dR}(A\cup B) \to \dots \to H^{n-1}_{dR}(A\cap B) \to H^n_{dR}(A\cup B) \to H^n_{dR}(A)\oplus H^n_{dR}(B) \to H^n_{dR}(A\cap B) \to 0$$

Proof: To construct the sequence, we first find a short exact sequence

$$0 \to \Omega^p(A\cup B) \to \Omega^p(A)\oplus\Omega^p(B) \to \Omega^p(A\cap B) \to 0$$

for each p. Specifically, if $i_A : A \to A\cup B$, $i_B : B \to A\cup B$, $j_A : A\cap B \to A$, & $j_B : A\cap B \to B$ are the obvious inclusions, then the sequence is:

$$0 \longrightarrow \Omega^p(A\cup B) \xrightarrow{i_A^*\oplus i_B^*} \Omega^p(A)\oplus\Omega^p(B) \xrightarrow{(j_A^*,\,-j_B^*)} \Omega^p(A\cap B) \to 0.$$

To see that this is exact, note that:

① $i_A^*\oplus i_B^* : \Omega^p(A\cup B) \to \Omega^p(A)\oplus\Omega^p(B)$ is obviously injective.

② $\ker(j_A^*, -j_B^*) \supset \operatorname{im}(i_A^*\oplus i_B^*)$: Clearly $(j_A^*, -j_B^*)\circ(i_A^*\oplus i_B^*)\,\omega = j_A^* i_A^*\omega - j_B^* i_B^*\omega = \omega|_{A\cap B} - \omega|_{A\cap B} = 0$,

so $\ker(j_A^*, -j_B^*) = \operatorname{im}(i_A^*\oplus i_B^*)$

OTOH, if $(\omega,\eta)\in\Omega^p(A)\oplus\Omega^p(B)$ s.t. $j_A^*\omega - j_B^*\eta = 0$, then

$j_A^*\omega = j_B^*\eta \Leftrightarrow \omega|_{A\cap B} = \eta|_{A\cap B}$.

Then define $\theta := \begin{cases} \omega & \text{on } A \\ \eta & \text{on } B \end{cases}$. Since they agree on the overlap θ is well-defined & $(\omega,\eta) = (i_A^*\oplus i_B^*)\theta$, so we get the other containment.

③ $j_A^* - j_B^*$ is surjective: Suppose $\omega\in\Omega^p(A\cap B)$. Let $\{\varphi_A, \varphi_B\}$ be a partition of unity subordinate to the cover $\{A, B\}$ of $A\cup B$. Then $(\varphi_B\omega, -\varphi_A\omega)\in\Omega^p(A)\oplus\Omega^p(B)$ and

$(j_A^* - j_B^*)(\varphi_B\omega, -\varphi_A\omega) = \varphi_B\omega + \varphi_A\omega = \omega$ on $A\cap B$.

Now, we recall that our spaces of p-forms fit into their own chain complexes, so we get a diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \Omega^p(A\cup B) \arrow[r, "i_A^*\oplus i_B^*"] \arrow[d, "d"] & \Omega^p(A)\oplus\Omega^p(B) \arrow[r, "j_A^*-j_B^*"] \arrow[d, "d"] & \Omega^p(A\cap B) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{p+1}(A\cup B) \arrow[r, "i_A^*\oplus i_B^*"] \arrow[d, "d"] & \Omega^{p+1}(A)\oplus\Omega^{p+1}(B) \arrow[r, "j_A^*-j_B^*"] \arrow[d, "d"] & \Omega^{p+1}(A\cap B) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{p+2}(A\cup B) \arrow[r, "i_A^*\oplus i_B^*"] & \Omega^{p+2}(A)\oplus\Omega^{p+2}(B) \arrow[r, "j_A^*-j_B^*"] & \Omega^{p+2}(A\cap B) \arrow[r] & 0
\end{tikzcd}
$$

Then to get the map $H^p_{dR}(A\cap B)\to H^{p+1}_{dR}(A\cup B)$ that we need for the Mayer-Vietoris sequence (called the connecting homomorphism) we need a map from closed p-forms on $A\cap B$ to closed $(p+1)$-forms on $A\cup B$. This will be a version of the Snake Lemma.

The proof is a diagram chase...

Suppose $\omega\in\Omega^p(A\cap B)$ is closed. Then $d\omega=0$ in $\Omega^{p+1}(A\cap B)$. Now, since $j_A^*-j_B^*$ is surjective, $\exists$

$(\alpha,\beta)\in\Omega^p(A)\oplus\Omega^p(B)$ s.t. $(j_A^*-j_B^*)(\alpha,\beta)=\omega$.

Now consider $(d\alpha,d\beta)\in\Omega^{p+1}(A)\oplus\Omega^{p+1}(B)$. Since the diagram commutes, we see that

$$0=d\omega=d(j_A^*-j_B^*)(\alpha,\beta)=(j_A^*-j_B^*)\circ(d\oplus d)(\alpha,\beta)=(j_A^*-j_B^*)(d\alpha,d\beta),$$

so $(d\alpha,d\beta)\in\ker(j_A^*-j_B^*)$. Therefore, since the middle row is exact, $(d\alpha,d\beta)\in\operatorname{im}(i_A^*\oplus i_B^*)$, so $\exists\ \eta\in\Omega^{p+1}(A\cup B)$ s.t. $(d\alpha,d\beta)=(i_A^*\eta,\ i_B^*\eta)$. Moreover, since $(i_A^*\oplus i_B^*)$ is injective, η is unique.

Then we will define the connecting homomorphism $\delta^*: H^p_{dR}(A\cap B)\to H^{p+1}_{dR}(A\cup B)$ by

$$[\omega]\mapsto[\eta]$$

To see that this is well-defined, we need to know that: ① η is closed

② $[\eta]$ is independent of the choice of representative ω of $[\omega]$.

③ $[\eta]$ is ind. of choice of α,β.

For ①, note that the diagram commutes, so

$$(i_A^*\oplus i_B^*)\circ d\eta = (d\oplus d)\circ(i_A^*\oplus i_B^*)\eta = (d\oplus d)(d\alpha, d\beta) = (dd\alpha, dd\beta) = 0.$$

But since the bottom row is exact, $i_A^*\oplus i_B^*$ is injective, so we see that $d\eta = 0$.

For ②, suppose ω' is another rep. of $[\omega]$. Then $\omega' = \omega + d\gamma$ for some $\gamma\in\Omega^{p-1}(A\cap B)$. $(j_A^*-j_B^*)$ is surjective, so $\exists$

$(\alpha_1,\beta_1)\in\Omega^{p-1}(A)\oplus\Omega^{p-1}(B)$ s.t. $(j_A^*-j_B^*)(\alpha_1,\beta_1)=\gamma$.

Then $(\alpha+d\alpha_1, \beta+d\beta_1)\longmapsto \omega'=\omega+d\gamma$ and we see that

$(d(\alpha+d\alpha_1), d(\beta+d\beta_1)) = (d\alpha, d\beta)$, so the elt $(d\alpha,d\beta)\in\Omega^{p+1}(A)\oplus\Omega^{p+1}(B)$ is ind. of ω.

For ③, suppose (α',β') is another preimage of ω. Then

$$(\alpha'-\alpha, \beta'-\beta)\in\ker(j_A^*-j_B^*) = \operatorname{im}(i_A^*\oplus i_B^*),$$

so $(\alpha'-\alpha,\beta'-\beta) = (i_A^*\xi, i_B^*\xi)$ for some $\xi\in\Omega^p(A\cup B)$. Then since the diagram commutes,

$$(i_A^*\oplus i_B^*)\circ d\xi = (d\oplus d)\circ(i_A^*\oplus i_B^*)\xi = (d\oplus d)(\alpha'-\alpha,\beta'-\beta) = (d\alpha'-d\alpha, d\beta'-d\beta).$$

Therefore, $(i_A^*\oplus i_B^*)(\eta+d\xi) = (d\alpha', d\beta')$, and indeed $[\eta+d\xi]=[\eta]$.

Putting this all together, then, we have the sequence

$$\cdots\to H^p_{dR}(A\cap B)\xrightarrow{\partial^*} H^{p+1}_{dR}(A\cup B)\xrightarrow{i_A^*\oplus i_B^*} H^{p+1}_{dR}(A)\oplus H^{p+1}_{dR}(B)\xrightarrow{j_A^*-j_B^*} H^{p+1}_{dR}(A\cap B)\xrightarrow{\partial^*} H^{p+2}_{dR}(A\cup B)\to\cdots$$

All that remains to check is that the sequence is exact. In fact, only need to check exactness at ∂^*, since exactness elsewhere follows from the short exact sequences.

Now certainly $\operatorname{im}\partial^*\subset\ker(i_A^*\oplus i_B^*)$ by how we defined ∂. OTOH, if $[\eta]\in\ker(i_A^*\oplus i_B^*)$, meaning that $(i_A^*\eta, i_B^*\eta) = (d\alpha, d\beta)$ for some $(\alpha,\beta)\in\Omega^p(A)\oplus\Omega^p(B)$, then $[\eta]=\partial^*[j_A^*\alpha - j_B^*\beta]$, so $\operatorname{im}\partial^*\supset\ker(i_A^*\oplus i_B^*)$.

Finally, if $[\omega]=(j_A^*-j_B^*)([\alpha],[\beta])$, then $\omega = j_A^*\alpha - j_B^*\beta$ for closed α & β, so $(d\alpha,d\beta)=0 \Rightarrow 0=[\eta]=\partial^*[\omega]$

and so $\operatorname{im}(j_A^\# - j_B^\#) \subset \ker \partial^\#$. OTOH, if $\partial^\#[\omega] = 0$, then $\partial\omega = \eta = d\xi$ for some $\xi \in \Omega^p(A \cup B)$. Then

$$(d\alpha, d\beta) = (i_A^\# \oplus i_B^\#) d\xi = (d \oplus d)(i_A^\# \oplus i_B^\#)\xi = (d i_A^\# \xi, d i_B^\# \xi).$$

Therefore, $(d(\alpha - i_A^\# \xi), d(\beta - i_B^\# \xi)) = 0$ and

$$(j_A^\# - j_B^\#)(\alpha - i_A^\# \xi, \beta - i_B^\# \xi) = j_A^\# \alpha - j_B^\# \beta = \omega,$$

so $[\omega] = (j_A^\# - j_B^\#)([\alpha - i_A^\# \xi], [\beta - i_B^\# \xi])$ and we see $\ker \partial^\# \subset \operatorname{im}(j_A^\# - j_B^\#)$. ∎