Math 8250: Day 22

Recall the

Poincaré Lemma: Suppose M is a mfld & that $\pi: M\times\mathbb{R}\to M$ is the obvious map. If $i_a: M\to M\times\mathbb{R}$ is as above, then

$$i_a^*: H^p_{dR}(M\times\mathbb{R})\to H^p_{dR}(M) \qquad \& \qquad \pi^*: H^p_{dR}(M)\to H^p_{dR}(M\times\mathbb{R})$$

are inverses of each other. As a consequence, both are isomorphisms.

The key to proving this are propositions 1, 2, & 3 from last time

Prop 1: Let $\varphi: V\to U$ be a diffeo. of open subsets of $\mathbb{R}^n$ and let $\Phi: V\times\mathbb{R}\to U\times\mathbb{R}$ be the diffeo. $\Phi=\varphi\times id_{\mathbb{R}}$. Then

$$\Phi^* h_p\omega = h_p\Phi^*\omega$$

for any $\omega\in\Omega^p(U\times\mathbb{R})$.

This will then allow us, just as when we were defining integration, to define h_p as an operator on manifolds.

Proof: Write ω in local coords:

$$\omega=\sum_I f_I(x,t)\,dt\wedge dx_I+\sum_J g_J(x,t)\,dx_J.$$

Then

$$\Phi^*h_p\omega=\Phi^*\Big(\sum_I\Big(\int_0^t f_I(x,s)\,ds\Big)dx_I\Big)=\sum_I\Big(\int_0^t f_I\circ\Phi(x,s)\,ds\Big)\Phi^*dx_I$$

$$=\sum_I\Big(\int_0^t f_I(\varphi(x),s)\,ds\Big)\varphi^*dx_I.$$

OTOH,

$$h_p\Phi^*\omega=h_p\Big(\sum_I(f_I\circ\Phi)(x,t)\,\Phi^*dt\wedge\Phi^*dx_I+\sum_J(g_J\circ\Phi)(x,t)\,\Phi^*dx_J\Big)$$

$$=h_p\Big(\sum_I f_I(\varphi(x),t)\,dt\wedge\varphi^*dx_I+\sum_J g_J(\varphi(x),t)\,\varphi^*dx_J\Big)$$

$$=\sum_I\Big(\int_0^t f_I(\varphi(x),s)\,ds\Big)\varphi^*dx_I.$$

So indeed $\Phi^*h_p\omega=h_p\Phi^*\omega$. ∎

Prop 3: Let $\pi: M\times\mathbb{R}\to M$ be the projection & let $i_a: M\to M\times\mathbb{R}$ be given by $i_a(x)=(x,a)$. Then

$$dh_p\omega + h_{p+1}d\omega = \omega - \pi^* i_a^* \omega$$

for all $\omega\in\Omega^p(M\times\mathbb{R})$.

Proof: HW. ∎

Then the big consequence of the Poincaré lemma is

Thm: Suppose M & N are manifolds & $f,g: M\to N$ are homotopic smooth maps. Then as maps $H^p_{dR}(N)\to H^p_{dR}(M)$,

$$f^\# = g^\#.$$

Cor: If M is contractible, then $H^p_{dR}(M)=0$ for all $p>0$.

Proof: Obviously true when M is a point, so then apply the theorem. ∎

Here's another nice, purely form-theoretic result:

Thm: For any $n>0$,

$$H^p_{dR}(S^n)\cong\begin{cases}\mathbb{R} & \text{if } p=0 \text{ or } n\\ 0 & \text{otherwise}\end{cases}$$

We have shown that $H^0_{dR}(S^n)\cong\mathbb{R}$ for all n & that $H^1_{dR}(S^1)\cong\mathbb{R}$. Moreover, your work in HWS #2 easily generalizes to show that $H^1_{dR}(S^n)=0$ for all $n>1$.

We will combine these results w/ an inductive application of the following Mayer-Vietoris type result:

Prop: Let $n>1$ & let $U_1=S^n-\{\text{south pole}\}$ & $U_2=S^n-\{\text{north pole}\}$. Then

$$H^p_{dR}(U_1\cup U_2)\cong H^{p-1}_{dR}(U_1\cap U_2)$$

for any $p>1$.

Clearly $U_1 \cup U_2 = S^n$, whereas $U_1 \cap U_2 \cong S^{n-1} \times \mathbb{R}$. Therefore, since the Poincaré lemma implies $H^{p-1}_{dR}(S^{n-1} \times \mathbb{R}) \cong H^{p-1}_{dR}(S^{n-1})$,

we see that $H^p_{dR}(S^n) \cong H^{p-1}_{dR}(S^{n-1})$.

Proof of Prop: Let $j_i : U_i \to S^n$ be the inclusion.

We construct a map $\varphi : H^p_{dR}(U_1 \cup U_2) \to H^{p-1}_{dR}(U_1 \cap U_2)$ as follows: let $[\omega] \in H^p_{dR}(U_1 \cup U_2)$ & let ω be any representative.

Then $\omega \in \Omega^p(U_1 \cup U_2)$ is closed, and so $j_i^* \omega \in \Omega^p(U_i)$ is closed. Since U_i is contractible, $H^p_{dR}(U_i) = 0$, so

the closed form $j_i^* \omega$ must be exact. Let γ_i be a primitive; i.e., $d\gamma_i = j_i^* \omega$. Then define the form

$\eta \in \Omega^{p-1}(U_1 \cap U_2)$ by

$$\eta := \gamma_1 - \gamma_2 .$$

Then let $\varphi([\omega]) = [\eta]$. Many things to check:

① η is closed: Clearly $d\gamma_1|_{U_1 \cap U_2} = j_i^* \omega|_{U_1 \cap U_2} = d\gamma_2|_{U_1 \cap U_2}$, so $d\eta = d\gamma_1 - d\gamma_2 = 0$.

② Despite ambiguity in choice of γ_i, $[\eta]$ is well-defined:

Suppose $\tilde{\gamma}_i$ are alternative primitives for $j_i^* \omega$. Then $d\tilde{\gamma}_i = d\gamma_i$, so

$\tilde{\gamma}_i - \gamma_i$ is closed $\Rightarrow$ $\tilde{\gamma}_i - \gamma_i$ is exact $\Rightarrow$ $\tilde{\gamma}_i = \gamma_i + d\xi_i$, so

$\tilde{\gamma}_1 - \tilde{\gamma}_2 = (\gamma_1 + d\xi_1) - (\gamma_2 + d\xi_2) = (\gamma_1 - \gamma_2) + d(\xi_1 - \xi_2)$, so

$[\tilde{\gamma}_1 - \tilde{\gamma}_2] = [\gamma_1 - \gamma_2] = [\eta]$.

③ If $\omega \in \Omega^p(S^n)$ is exact, then the corresponding η reps. 0 in $H^{p-1}_{dR}(U_1 \cap U_2)$:

If $\omega = d\gamma$, then we can choose $\gamma_1 = j_1^* \gamma$ & $\gamma_2 = j_2^* \gamma$, so $\eta = \gamma_1 - \gamma_2 = (j_1^* \gamma - j_2^* \gamma)|_{U_1 \cap U_2} = 0$

Now, we can construct a map sending closed $(p-1)$-forms on $U_1 \cap U_2$ to closed p-forms on $U_1 \cup U_2$ which is the inverse of the above map at the level of cohomology:

Let ψ_1, ψ_2 be fctns on S^n s.t. ψ_1 vanishes in a nbhd of the north pole, ψ_2 vanishes in a nbhd of the south pole, and

$\psi_1 + \psi_2 = 1$ everywhere.

If $\eta \in \Omega^{p-1}(U_1 \cap U_2)$ is closed, define $\gamma_1 = \psi_1 \eta$ on U_1 (although η might blow up at the pole, the vanishing of

ψ_1 kills that off) and let $\gamma_2 = -\psi_2 \eta$ on U_2.

Then $\gamma_1 - \gamma_2 = \eta$ on all of $U_1 \cap U_2$. Define $\omega \in \Omega^p(U_1 \cup U_2)$ by setting $\omega|_{U_1} = d\gamma_1$ & $\omega|_{U_2} = d\gamma_2$.

Then $d\gamma_1 - d\gamma_2 = d(\gamma_1 - \gamma_2) = d\eta = 0$ on $U_1 \cap U_2$, so the form ω is well-defined &, in fact, closed.

This procedure clearly gives an inverse to φ at the level of cohomology, so we see that

$$\varphi : H^p_{dR}(U_1 \cup U_2) \to H^{p-1}_{dR}(U_1 \cap U_2)$$

is an isomorphism, as desired. $\blacksquare$