Math 8250: Day 21

de Rham Theorem: There exists a linear map

$$H^p_{dR}(M) \to H_p(M;\mathbb{R})^*$$

which is a vector space isomorphism.

We're not ready to prove this yet, but the map is given by: if $[\alpha] \in H^p_{dR}(M)$, then $[\alpha]([z]) = \int_z \alpha$ for any representatives α of $[\alpha]$ and z of $[z]$.

Prop: For any mfld M, $H^0_{dR}(M) \cong \mathbb{R}^d$, where d is the # of connected components of M.

Proof: There are no exact 0-forms, so we just need to see that the space of closed 0-forms is d-dimensional.

Suppose $f \in \Omega^0(M)$ s.t. $df = 0$. Then if $p \in M$ & $x_1, \ldots, x_n$ are local coords. in a nbhd U of p, then

$$0 = df|_U = \sum \frac{\partial f}{\partial x_i}\, dx_i .$$

But clearly this means each $\frac{\partial f}{\partial x_i} = 0$, so f is constant on U. Therefore, f is a smooth, locally constant function on M, so is constant on each connected component of M. Therefore, closed 0-forms are locally constant, so form a d-dim'l v.s. $\blacksquare$

Here is a theorem we will spend the rest of the week proving:

Thm: Suppose M & N are manifolds & $f, g: M \to N$ are homotopic smooth maps. Then as maps $H^p_{dR}(N) \to H^p_{dR}(M)$,

$$f^\# = g^\#.$$

Now, let's think about what f & g being homotopic means. It means $\exists\, H: M \times \mathbb{R} \to N$ s.t. $H(x,0) = f(x)$ & $H(x,1) = g(x)$.

For any $a \in \mathbb{R}$, let $i_a: M \to M \times \mathbb{R}$ be given by $i_a(x) = (x,a)$. Then notice that

$$f(x) = H(x,0) = (H \circ i_0)(x) \qquad \& \qquad g(x) = H(x,1) = (H \circ i_1)(x),$$

so $f^\# = (H \circ i_0)^\# = i_0^\# \circ H^\#$ & $g^\# = (H \circ i_1)^\# = i_1^\# \circ H^\#$.

Then the fact that $f^\# = g^\#$ will follow if we know that $i_0^\# = i_1^\#$. This, in turn, will be a consequence of the

Poincaré Lemma: Suppose M is a mfd & that $\pi : M\times\mathbb{R}\to M$ is the obvious map. If $i_a : M\to M\times\mathbb{R}$ is as above, then

$$i_a^{\#} : H^p_{dR}(M\times\mathbb{R}) \to H^p_{dR}(M) \qquad \& \qquad \pi^{\#} : H^p_{dR}(M)\to H^p_{dR}(M\times\mathbb{R})$$

are inverses of each other. As a consequence, both are isomorphisms.

Corollary: $H^p(\mathbb{R}^n)=0$ for all $p>0$.

Pf: $H^p(\{0\})=0$ for $p>0$. Now use the Poincaré Lemma inductively. ▮

The Poincaré Lemma implies that $i_0^{\#}=i_1^{\#}$ since both are inverses of the same map $\pi^{\#}$. So once we have proved the Poincaré Lemma we'll have proved the theorem.

Half of the Poincaré Lemma is trivial: clearly, $\pi\circ i_a = id_M$, so $id_{H(M)} = id_M^{\#} = (\pi\circ i_a)^{\#} = i_a^{\#}\circ\pi^{\#}$, and so $i_a^{\#}$ is a left inverse for $\pi^{\#}$. Of course, the other way is trickier. What I want to show is that $\pi^{\#}\circ i_a^{\#} = id^{\#}_{M\times\mathbb{R}}$.

In general, whenever we have maps $F,G : X\to Y$ b/w mfds & want to show $F^{\#}=G^{\#}$, the strategy is the following:

Consider $F^*, G^* : \Omega^p(Y)\to\Omega^p(X)$. Then it suffices to show that for any closed $\omega\in\Omega^p(Y)$,

$(F^*-G^*)\omega = F^*\omega - G^*\omega$ represents $[0]$ in $H^p_{dR}(X)$, which is to say

(*) $F^*\omega - G^*\omega = d\gamma_\omega$ for some $\gamma_\omega\in\Omega^{p-1}(X)$ which obviously depends on ω.

In fact, we want to show, slightly more generally, that, as maps $\Omega^p(Y)\to\Omega^p(X)$,

(**) $$F^*-G^* = h_{p+1}\circ d + d\circ h_p$$

for some collection of maps $\{h_k\}$ where $h_k : \Omega^k(Y)\to\Omega^{k-1}(X)$. The collection $\{h_k\}$ is called a **homotopy operator** for F & G. Clearly, if $\omega\in\Omega^p(Y)$ is closed, then (**) will imply that

$$F^*\omega - G^*\omega = h_{p+1}\circ \overset{0}{\cancel{d\omega}} + d\circ h_p\,\omega = d(h_p\,\omega).$$

Letting $\gamma_\omega := h_p\,\omega$ yields (*).

In our case, if we let $F = id_{M\times\mathbb{R}}$ & $G = \pi \circ i_a$, then life is slightly easier as $Y = X$ and we're just looking for an operator

$h_p : \Omega^p(M\times\mathbb{R}) \to \Omega^{p-1}(M\times\mathbb{R})$ s.t. $dh_p\omega + h_{p+1}d\omega = \omega - \pi^* i_a^* \omega \quad \forall\, \omega \in \Omega^p(M\times\mathbb{R})$.

Okay, well, here is the operator, which we first define when $M = U \subset \mathbb{R}^n$ is an open set:

any p-form on $U\times\mathbb{R}$ can be written as

$$\omega = \sum_I f_I(x,t)\, dt \wedge dx_I + \sum_J g_J(x,t)\, dx_J$$

where t is the coord. on $\mathbb{R}$, $x_1, \ldots, x_n$ are the coords on $\mathbb{R}^n$, the first sum is over all length $p-1$ multiindices & the second is over all length p multiindices. Define h_p by

$$h_p\omega := \sum_I \left(\int_0^t f_I(x,s)\, ds \right) dx_I .$$

Prop 1: Let $\varphi : V \to U$ be a diffeo. of open subsets of $\mathbb{R}^n$ and let $\Phi : V\times\mathbb{R} \to U\times\mathbb{R}$ be the diffeo. $\Phi = \varphi \times id_{\mathbb{R}}$. Then

$$\Phi^* h_p \omega = h_p \Phi^* \omega$$

for any $\omega \in \Omega^p(U\times\mathbb{R})$.

This will then allow us, just as when we were defining integration, to define h_p as an operator on manifolds:

Prop 2: $\exists!$ operator $h_p : \Omega^p(M\times\mathbb{R}) \to \Omega^{p-1}(M\times\mathbb{R})$ defined for all mflds M so that

1. If $\varphi : M \to M'$ is a diffeo. & $\Phi = \varphi \times id_{\mathbb{R}} : M\times\mathbb{R} \to M'\times\mathbb{R}$, then $\Phi^* h_p = h_p \Phi^*$.
2. If $M \subset \mathbb{R}^n$ is an open set, then h_p is as defined above.

Then w/ this operator defined, the key identity to complete the proof of the Poincaré Lemma is

Prop 3: Let $\pi : M\times\mathbb{R} \to M$ be the projection & let $i_a : M \to M\times\mathbb{R}$ be given by $i_a(x) = (x,a)$. Then

$$dh_p\omega + h_{p+1}d\omega = \omega - \pi^* i_a^* \omega$$

for all $\omega \in \Omega^p(M\times\mathbb{R})$.

To be continued...