## This week: de Rham cohomology

Short version: on a (cpct) mfd $M^n$ recall the chain cplx

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \cdots \xrightarrow{d_{n-2}} \Omega^{n-1}(M) \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0$$

We define the $k^{th}$ de Rham cohomology group of $M$ to be

$$H^k_{dR}(M) := \ker d_k / \operatorname{im} d_{k-1}$$

b/c $d^2=0$, we know $\operatorname{im} d_{k-1} \subset \ker d_k$.

Consider $M=S^1$. On HW you showed that: ① $d\theta = \frac{1}{x^2+y^2}(-y\,dx + x\,dy)$ is closed & not exact

② $\forall\, \omega \in \Omega^1(S^1)$, $\omega - c\,d\theta$ is exact, where $c = \frac{1}{2\pi}\int_{S^1}\omega$.

Hence, $d\theta$ represents a nontrivial elt. of $H^1_{dR}(S^1)$, and every 1-form is in the same class as $c\,d\theta$,

so $H^1_{dR}(S^1) \cong \mathbb{R}$... agreeing nicely w/ the fact that $H^1(S^1;\mathbb{R}) \cong \mathbb{R}$.

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Now, the de Rham cohomology groups turn out to be homotopy invariants of a mfd; in other words, homotopy-equivalent mfds must have the same de Rham cohomology groups. This will take some work to show, but it is easy to see that the de Rham groups are diffeomorphic invts:

If $f: M \to N$ is smooth, then $f$ induces a homomorphism $f^\#: H^k_{dR}(N) \to H^k_{dR}(M)$ for all $k$ by

$$f^\#([\alpha]) = [f^*\alpha]$$

for any representative $\alpha \in \Omega^k(N)$ of the class $[\alpha] \in H^k_{dR}(N)$.

To see that this is well-defined, note that, if $\alpha, \alpha'$ repsent the same class, then

$$\alpha' - \alpha = d\beta \text{ for some } \beta \in \Omega^{k-1}(N),$$

So

$$f^{\#}[\alpha'] = [f^{*}\alpha'] = [f^{*}(\alpha + d\beta)] = [f^{*}\alpha] + [f^{*}d\beta] = [f^{*}\alpha] + \underbrace{\cancel{[d f^{*}\beta]}}_{0} = [f^{*}\alpha] = f^{\#}[\alpha].$$

Moreover, if $g: N \to Y$ is also smooth, then $(g\circ f)^{\#}: H^k_{dR}(Y) \to H^k_{dR}(M)$ is simply

$$(g\circ f)^{\#} = f^{\#}\circ g^{\#}$$

since the corresponding statement about $(g\circ f)^{*}$ is true.

Therefore, if $f: M \to N$ is a diffeo, then $f^{\#}: H^k_{dR}(N) \to H^k_{dR}(M)$ has a left & right inverse, namely $(f^{-1})^{\#}$ since

$$(f^{-1})^{\#}\circ f^{\#} = (f\circ f^{-1})^{\#} = (id_N)^{\#} = id_{H^k_{dR}(N)} \quad \& \quad f^{\#}\circ (f^{-1})^{\#} = (f^{-1}\circ f)^{\#} = (id_M)^{\#} = id_{H^k_{dR}(M)}$$

So de Rham cohomology is indeed a diffeomorphism invariant.

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The key to showing that the de Rham cohomology groups are homotopy invariants is to show that

$$H^p_{dR}(M) \cong H^p(M;\mathbb{R}),$$

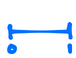
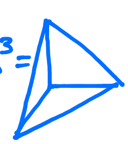
the $p^{th}$ singular cohomology group w/ real coefficients.

To build up to this let's first define the differentiable singular cohomology groups

Def: For $p \geq 1$, let

$$\Delta^p := \{(a_1,\dots,a_p) \in \mathbb{R}^p \mid \sum_{i=1}^{p} a_i \leq 1 \text{ and each } a_i \geq 0\}$$

be the standard p-simplex, and let $\Delta^0 = \{0\}$.

Ex: $\Delta^1 =$  , $\Delta^2 =$  , $\Delta^3 =$  , etc.

Def: If $M$ is a mfd, then a differentiable singular p-simplex in $M$ is a map $\sigma: \Delta^p \to M$ that extends to a smooth map of a nbhd of $\Delta^p$ in $\mathbb{R}^p$ into $M$.

Each simplex has a boundary which pictorially is the obvious thing but which is a notational mess:

Define $k_i^p : \Delta^p \to \Delta^{p+1}$ $\forall\ 0 \le i \le p+1$ by:

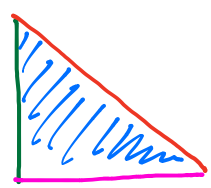
$p=0$: $k_0^0(0) = 1$, $k_1^0(0) = 0$

$$p \ge 1: \begin{cases} k_0^p(a_1, \dots, a_p) = (1 - \sum a_i, a_1, \dots, a_p) \\ k_i^p(a_1, \dots, a_p) = (a_1, \dots, a_{i-1}, 0, a_i, \dots, a_p) & \text{for } i \ge 1 \end{cases}$$

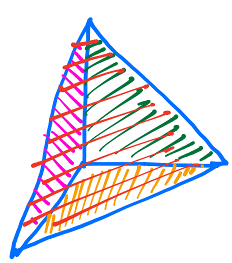
Then the $i^{th}$ face of a $p$-simplex $\sigma$ in $M$ is $\sigma^i := \sigma \circ k_i^{p-1}$ and the boundary of $\sigma$ is

$$\partial \sigma = \sum_{i=0}^{p} (-1)^i \sigma^i$$

Ex: Consider $\sigma = id : \Delta^2 \to \mathbb{R}^2$. Then $\partial \sigma = \sigma^0 - \sigma^1 + \sigma^2$



Consider $\sigma = id$ as a map $\Delta^3 \to \mathbb{R}^3$. Then $\partial \sigma = \sigma^0 - \sigma^1 + \sigma^2 - \sigma^3$



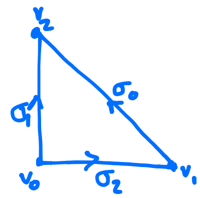
$\partial \sigma = \sum (-1)^i \sigma^i$ is an example of a $(p-1)$-chain on $M$. In general, a $p$-chain on $M$ is a finite linear combination

$$\sum c_i \sigma_i$$

where $a_i \in \mathbb{R}$ and the $\sigma_i$ are $p$-simplices.

Then we can define $C_p(M; \mathbb{R})$ to be the vector space of $p$-chains on $M$, called the $p^{th}$ singular chain group of $M$, and the boundary operator extends to a linear map $\partial : C_p(M;\mathbb{R}) \to C_{p-1}(M;\mathbb{R})$. Then it's an annoying but essentially trivial calculation to see that $\partial_p \circ \partial_{p+1} = 0$.

Ex: Again let $\sigma = id : \Delta^2 \to \mathbb{R}^2$. Then $\partial_1 \circ \partial_2 \sigma = \partial_1(\sigma^0 - \sigma^1 + \sigma^2)$




$$= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0)$$
$$= 0$$

Therefore, we get a chain complex

$$\cdots \longrightarrow C_n(M;\mathbb{R}) \xrightarrow{\partial_n} C_{n-1}(M;\mathbb{R}) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_1(M;\mathbb{R}) \xrightarrow{\partial_1} C_0(M;\mathbb{R}) \xrightarrow{\partial_0} \cdots$$

and can define the $p^{th}$ singular homology group $H_p(M;\mathbb{R}) := \ker \partial_p / \operatorname{im} \partial_{p+1}$.

de Rham Theorem: There exists a linear map

$$H^p_{dR}(M) \to H_p(M;\mathbb{R})^*$$

which is a vector space isomorphism.