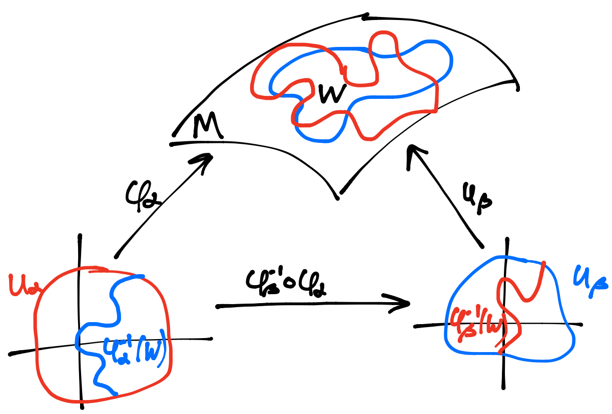
Math 8250: Day 2

Def: A *differentiable manifold of dimension* n is a Hausdorff, second countable topological space M together w/ a family of injective maps $\varphi_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ s.t.:

① $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$

② for any α, β s.t. $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W \neq \emptyset$, the sets $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the maps $\varphi_\beta^{-1} \circ \varphi_\alpha$ & $\varphi_\alpha^{-1} \circ \varphi_\beta$ are smooth (C^∞).

③ The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal w.r.t. ① & ②



The $(U_\alpha, \varphi_\alpha)$ are called *coordinate charts*

Ex: ① The maximal collection containing $(\mathbb{R}^n, id)$ obviously makes $\mathbb{R}^n$ into an n-manifold.

② $\mathbb{RP}^n$ is the set of straight lines in $\mathbb{R}^{n+1}$ or, equivalently, the quotient of $\mathbb{R}^n - \{0\}$ by the equiv. relation

$$(x_1, \dots, x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1}) \ \forall \lambda \in \mathbb{R}, \lambda \neq 0.$$

Write points in homogeneous coords $[x_1, \dots, x_n]$. In particular, if $x_i \neq 0$

$$[x_1, \dots, x_{n+1}] = \left[\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, x_n\right]$$

Define $\varphi_i : \mathbb{R}^n \to \mathbb{RP}^n$ by $\varphi_i(x_1, \dots, x_n) = [x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n]$

then maximal family containing $\{(\mathbb{R}^n, \varphi_i)\}$ makes $\mathbb{RP}^n$ into a manifold.

① Obvious

② $\varphi_i^{-1}(\varphi_i(\mathbb{R}^n) \cap \varphi_j(\mathbb{R}^n)) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j \neq 0\}$ if $i > j$, which is clearly open, and

$$\varphi_i^{-1} \circ \varphi_j(x_1, \ldots, x_n) = \varphi_i^{-1}[x_1, \ldots, x_{j-1}, 1, x_j, \ldots, x_n]$$

$$= \varphi_i^{-1}\left[\frac{x_1}{x_{i-1}}, \ldots, \frac{x_{j-1}}{x_{i-1}}, \frac{1}{x_{i-1}}, \frac{x_j}{x_{i-1}}, \ldots, \frac{x_{i-2}}{x_{i-1}}, 1, \frac{x_i}{x_{i-1}}, \ldots, \frac{x_n}{x_{i-1}}\right]$$

$$= \left(\frac{x_1}{x_{i-1}}, \ldots, \frac{x_{j-1}}{x_{i-1}}, \frac{1}{x_{i-1}}, \frac{x_j}{x_{i-1}}, \ldots, \frac{x_{i-2}}{x_{i-1}}, \frac{x_i}{x_{i-1}}, \ldots, \frac{x_n}{x_{i-1}}\right)$$

So $\varphi_i^{-1} \circ \varphi_j$ is clearly smooth.

Def: Let M^m, N^n be mflds. A cont. map $f: M \to N$ is *differentiable at* $p \in M$ if, given a chart $\psi: V \subset \mathbb{R}^n \to N$ containing $f(p)$ $\exists$ chart $\varphi: U \subset \mathbb{R}^m \to M$ at p s.t. $f(\varphi(U)) \subset \psi(V)$ and

$$\psi^{-1} \circ f \circ \varphi : U \subset \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable at $\varphi^{-1}(p)$. The map f is diff'ble on an open set if it is diff'ble at all pts in the set.

Tangent Vectors

To define tangent vectors on a mfld, we need to think about vectors in $\mathbb{R}^n$ in a slightly diff't way than you might be used to. Specifically, we're going to identify a vector $v \in \mathbb{R}^n$ w/ the operator on differentiable (or smooth) functions which gives the directional derivative with respect to v of the function.

More precisely, if $\alpha: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a smooth map (i.e., a smooth curve) with $\alpha(0) = p$, then write

$$\alpha(t)=(x_1(t),\dots,x_n(t)),\quad t\in(-\varepsilon,\varepsilon),\quad (x_1,\dots,x_n)\in\mathbb{R}^n$$

Then $\alpha'(0)=(x_1'(0),\dots,x_n'(0))=v\in\mathbb{R}^n$. Suppose f is differentiable in a nbhd of p. Then, restricting f to the image of α, the directional derivative w.r.t. v of f is

$$\left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0}=\sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{t=0}\left.\frac{dx_i}{dt}\right|_{t=0}=\left(\sum_i x_i'(0)\frac{\partial}{\partial x_i}\right)f.$$

Notice that this operator giving the directional deriv. w.r.t. v depends uniquely on v and is a linear derivation meaning that ① $v(f+\lambda g)=v(f)+\lambda v(g)$ $\forall f,g$ diff'ble in a nbhd of p & $\lambda\in\mathbb{R}$

② $v(fg)=f(p)v(g)+g(p)v(f)$

Df: Let M^n be a mfd. A smooth function $\alpha:(-\varepsilon,\varepsilon)\to M$ is a (smooth) curve in M. Suppose $\alpha(0)=p\in M$ and let $\mathcal{D}$ be the set of functions on M which are differentiable at p. The tangent vector to the curve α at $t=0$ is a function $\alpha'(0):\mathcal{D}\to\mathbb{R}$ given by

$$\alpha'(0)f=\left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0}$$

A tangent vector at p is the tangent vector at $t=0$ of some curve $\alpha:(-\varepsilon,\varepsilon)\to M$ w/ $\alpha(0)=p$. The set of all tangent vectors at p is the tangent space to M at p, denoted T_pM.

If (U,φ) is a coord. chart at the point p s.t. $p=\varphi(0)$, then

$$f\circ\varphi(q)=f(x_1,\dots,x_n),\quad q=(x_1,\dots,x_n)\in U$$

and

$$\varphi^{-1}\circ\alpha(t)=(x_1(t),\dots,x_n(t))$$

so

$$\alpha'(0)f=\left.\frac{d}{dt}(f\circ\alpha)\right|_{t=0}=\left.\frac{d}{dt}f(x_1(t),\dots,x_n(t))\right|_{t=0}$$

$$=\sum_{i=1}^n x_i'(0)\frac{\partial f}{\partial x_i}=\left(\sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_{\vec{0}}\right)f$$

In other words, we can write the vector $\alpha'(0)$ in local coords as

$$\alpha'(0) = \sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_0$$

The $\left(\frac{\partial}{\partial x_i}\right)_0$ give the basis of T_pM associated to the chart (U,φ).

Def: The tangent bundle of a manifold M^n, denoted TM, is the union of the tangent spaces:

$$TM := \bigcup_{p\in M} T_pM$$