Math 8250: Day 19

One more comment about volume forms: we've talked before about the fact that on a Riemannian manifold (M^n, g), there's a correspondence b/w vector fields & $(n-1)$-forms. Specifically, if $V \in \mathfrak{X}(M)$, then let $\alpha \in \Omega^1(M)$ s.t.

$$\alpha_p(V_p) = \|V_p\| \text{ and } \alpha_p(W_p) = 0 \quad \forall\, W_p \in T_pM \text{ s.t. } W_p \perp V_p$$

at each point $p \in M$. Obviously this depends on the metric. Then we get an $(n-1)$-form by taking the Hodge star:

$$\star_g \alpha \in \Omega^{n-1}(M).$$

Again, this depends on the metric.

However these dependencies on the metric cancel out, & in fact all that's needed is the volume form:

Given an oriented manifold M^n and a volume form $dvol_M$, we can associate an $(n-1)$-form to a vector field $V \in \mathfrak{X}(M)$ by

$$\omega := \iota_V \, dvol_M$$

If the volume form is a Riemannian volume form, then this is the same form we got before: let $x_1, \dots, x_n$ be ON local coords in a nbhd of $p \in M$ s.t. $V = a \frac{\partial}{\partial x_1}$. Then $\alpha = a\, dx_1$ and $\star \alpha = a\, dx_2 \wedge \dots \wedge dx_n$.

OTOH, $dvol_M = dx_1 \wedge \dots \wedge dx_n$ & so $\iota_V dvol_M = \iota_{a\frac{\partial}{\partial x_1}}(dx_1 \wedge \dots \wedge dx_n) = a\, dx_2 \wedge \dots \wedge dx_n = \star \alpha$.

The fact that we can canonically associate an $(n-1)$-form (but not a 1-form) to a vector field is often useful. For example:

Suppose M^3 is an embedded domain in $\mathbb{R}^3$. Define the Biot-Savart operator $BS: \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$BS(V)(y) := \frac{1}{4\pi} \int_M V(x) \times \frac{y-x}{|y-x|^3}\, dvol_x$$

Suppose V is a curl field on M (i.e. V is in the image of curl &, in particular, is divergence free) that's tangent to ∂M.

The example to keep in mind is when V is the magnetic field associated to a current flow.

Then it's a fact that $\nabla \times BS(V) = V$.

Now, for such vector fields we can define the helicity of the vector field by

$$H(V) := \frac{1}{4\pi}\int_{M\times M} V(x)\times V(y)\cdot\frac{x-y}{|x-y|^3}\,dvol_x\,dvol_y = \int_M V(y)\cdot BS(V)(y)\,dvol_y$$

Introduced by Woltjer, who noticed that $H(V)$ seems to be preserved under magnetohydrodynamic evolution when V is the magnetic field of a plasma (e.g. the Crab Nebula).

Now, the fact that helicity is an (approximate) invariant of ideal magnetohydrodynamics is a painful calculation... or a triviality if we translate into forms using the fact that ideal MHD evolution is by volume-preserving diffeos.

Arnol'd: Given $V\in\mathfrak{X}(M)$, define the associated 2-form $\omega := \iota_V\, dvol_M$. Then V being a curl field means ω is exact. (Remember the diagram

$$\begin{array}{ccccccc}
\Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \Omega^3(M)\\
\| & & \uparrow & & \uparrow & & \uparrow\\
C^\infty(M) & \xrightarrow{\nabla} & \mathfrak{X}(M) & \xrightarrow{\nabla\times} & \mathfrak{X}(M) & \xrightarrow{\nabla\cdot} & C^\infty(M)
\end{array}$$

)

So $\exists\ \alpha\in\Omega^1(M)$ s.t. $d\alpha=\omega$. Then $H(V)=\int_M V\cdot BS(V)\,dvol_M$ corresponds to

$$H(\omega) := \int_M \alpha\wedge d\alpha = \int_M \alpha\wedge\omega$$

which turns out to be independent of the choice of primitive α in suitably nice situations (like when M is simply-connected). So, as Arnol'd & also Cantarella-Parsley have pointed out, it's probably better to think of helicity as an invt. of 2-forms rather than of vector fields.

Now invariance under volume-preserving diffeomorphisms is easy. Suppose $f: M\to M'$ is a volume preserving diffeo. Then V gets pushed forward as $df\,V$, and so the corresponding 2-form is

$$\tilde\omega = \iota_{dfV}\,dvol_{M'}.$$

But, since f is volume-preserving, $dvol_{M'} = (f^{-1})^* dvol_M$:

$$\tilde{\omega}(v_2,v_3) = \iota_{df V}(f^{-1})^* d\mathrm{vol}_M(v_2,v_3) = (f^{-1})^* d\mathrm{vol}_M(df V, v_2, v_3) = d\mathrm{vol}_M(d(f^{-1})\,df V, d(f^{-1})v_2, d(f^{-1})v_3)$$

$$= d\mathrm{vol}_M(V, d(f^{-1})v_2, d(f^{-1})v_3)$$

$$= \iota_V d\mathrm{vol}_M(d(f^{-1})v_2, d(f^{-1})v_3)$$

$$= \omega(d(f^{-1})v_2, d(f^{-1})v_3)$$

$$= (f^{-1})^*\omega(v_2,v_3)$$

Moreover, $d(f^{-1})^*\alpha = (f^{-1})^* d\alpha = (f^{-1})^*\omega = \tilde{\omega}$, so

$$H(\tilde{\omega}) = \int_{M'} (f^{-1})^*\alpha \wedge \tilde{\omega} = \int_{M'} (f^{-1})^*\alpha \wedge (f^{-1})^*\omega = \int_{M'} (f^{-1})^*(\alpha\wedge\omega) = \int_M \alpha\wedge\omega = H(\omega).$$

So helicity is preserved under volume-preserving diffeos!

This tells you that helicity is a topological invariant of vector fields (still the only one known). In fact, it measures the "average asymptotic linking number" of orbits of the field.

One further comment: a unit vector field on S^3 (or really any 3-mfld) is equivalent to a map $f: S^3 \to S^2$ (b/c the tangent bundle is trivial).

So if $V \in \mathfrak{X}(S^3)$ corresponds to $f_V: S^3 \to S^2$ and ω is the std. area form on S^2 normalized s.t. $\int_{S^2}\omega = 1$, then

$$\omega_V = \iota_V d\mathrm{vol}_{S^3} = f_V^*\omega.$$

It turns out that ω_V is exact, so $\exists\, \alpha_V \in \Omega^1(S^3)$ s.t. $d\alpha_V = \omega_V$ (as we'll see, this is b/c $H^2(S^3;\mathbb{R}) = 0$)

$$H(\omega) := \int_{S^3} \alpha_V \wedge d\alpha_V = \int_{S^3} d^{-1}(\omega_V)\wedge\omega_V$$

But this is exactly Whitehead's 1947 integral formula for the Hopf invariant of f!