Math 8250: Day 18

Volume Forms

Recall that an n-manifold M is orientable $\Leftrightarrow$ $\exists$ nonvanishing $\omega \in \Omega^n(M)$, and ω defines the preferred orientation.

Def: If M^n is oriented, then a nonvanishing $\omega \in \Omega^n(M)$ which gives the preferred orientation is a volume form.

This then gives a way to integrate functions (rather than n-forms) over the mfld: if M^n has volume form $dVol_M$ and $f \in C^\infty(M)$, then we can define the integral of f over M to be $\int_M f\, dvol_M$.

Ex: If (M^{2n}, ω) is a symplectic mfld, then by definition $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$ is never zero, so it gives a volume form on M.

- Similarly, if (M^{2n+1}, α) is a contact mfld, then $\alpha \wedge (d\alpha)^n = \alpha \wedge d\alpha \wedge \cdots \wedge d\alpha$ is a volume form on M.
- In $\mathbb{R}^n$ the standard volume form is $dx_1 \wedge \cdots \wedge dx_n$.

As you might expect from the last example & the fact that volume forms allow you to integrate functions, volume forms are closely related to (signed) measures: if $U \subset M$ is an (oriented) open set (or, really, set containing an open set which is contained in the closure of that open set) then the volume of U is defined to be

$$vol(U) := \int_U dvol_M$$

Of course, in particular, $vol(M) := \int_M dvol_M$.

In the case of a symplectic mfld (M^{2n}, ω), the volume form ω^n yields the Liouville measure on M.

In the case of $\mathbb{R}^n$, the standard volume form yields Lebesgue measure.

Okay, but of course there are infinitely many possible volume forms on an oriented mfld (just multiply any one by a non-vanishing function & you get another one), so it's much more interesting to consider volume forms that come from interesting geometric structures.

Again, the symplectic & contact volume forms are examples of this.

Another important example is the Riemannian volume form:

Given a Riemannian mfd (M,g), the Riemannian metric gives rise to the Hodge star operator

$$\star_g : \Omega^k(M) \to \Omega^{n-k}(M)$$

by extending your work on HW2 #1e (usually the subscript g gets omitted).

Then the Riemannian volume form induced by g is the form $dvol_g := \star_g 1$, where 1 is the constant function $p \mapsto 1$ $\forall p \in M$.

For example, for any $f \in C^\infty(M)$,

$$\int_M f = \int_M f \, dvol_g = \int_M \star_g f.$$

If you like local coordinates, then, if $x_1,\dots,x_n$ are orthonormal local coords. on some chart U that agree w/ the orientation, then $dvol_g|_U = dx_1 \wedge \dots \wedge dx_n$. We can see that this is coord.-independent since, if $x_1',\dots,x_n'$ are ON coords on some overlapping chart U', then at $p \in U \cap U'$

$$(dx_1 \wedge \dots \wedge dx_n)_p = \det(A)\,(dx_1' \wedge \dots \wedge dx_n')_p$$

where $A \in SO(T_pM)$ sends e_i to e_i'; in particular, $\det(A) = 1$, so $dx_1 \wedge \dots \wedge dx_n = dx_1' \wedge \dots \wedge dx_n'$.

Ex: The area form on S^2.

Notice that the above implies that the Riemannian volume form $dvol_g$ is uniquely defined by the property that, at any $p \in M$ & for any oriented, ON basis $\{v_1,\dots,v_n\}$ for T_pM,

$$(dvol_g)_p(v_1,\dots,v_n) = 1.$$

On the sphere, it's a fact that if $\vec{v}, \vec{w} \in T_{\vec{p}} S^2$ form an oriented ON basis for $T_{\vec{p}} S^2$, then

$$\vec{p} \cdot (\vec{v} \times \vec{w}) = 1.$$

Therefore, it follows that the Riemannian volume form for the standard (round) metric on S^2 is given by

$$(d\mathrm{vol}_{S^2})_p (v, w) = p \cdot (v \times w)$$

In terms of coordinates, if $p = (x, y, z)$, then you can check that

$$(d\mathrm{vol}_{S^2})_p = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

Remark: If M^n is closed and $\omega, \eta \in \Omega^n(M^n)$ are volume forms that differ by an exact form (i.e., $\omega - \eta = d\xi$ for some $\xi \in \Omega^{n-1}(M)$), then

$$\int_M \omega = \int_M \eta$$

since $\int_M (\omega - \eta) = \int_M d\xi = 0$ by Stokes' Thm.

In fact, the converse is true:

Prop: If M^n is closed & oriented and $\omega, \eta \in \Omega^n(M)$ are volume forms s.t.

$$\int_M \omega = \int_M \eta$$

then $\exists\ \xi \in \Omega^{n-1}(M)$ s.t. $\omega = \eta + d\xi$.